Numerical Analyses for Electronagnetics

$$\nabla \times E + j\omega B = 0$$
$$\nabla \times H - j\omega D = J$$
$$\nabla \cdot D = \rho$$
$$\nabla \cdot B = 0$$

Ando & Hirokawa lab.

Takuichi Hirano (RA)

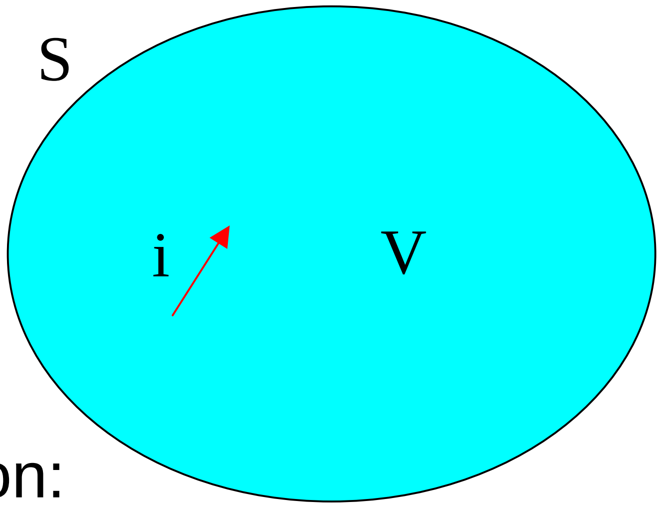
Differential Equations

Maxwell's Equation:

$$\nabla \times \mathbf{E} + j\omega\mu\mathbf{H} = 0$$

and

$$\nabla \times \mathbf{H} + j\omega\varepsilon\mathbf{E} = \mathbf{i}$$



Helmholtz's Equation:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$

or

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0$$

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{i}$$

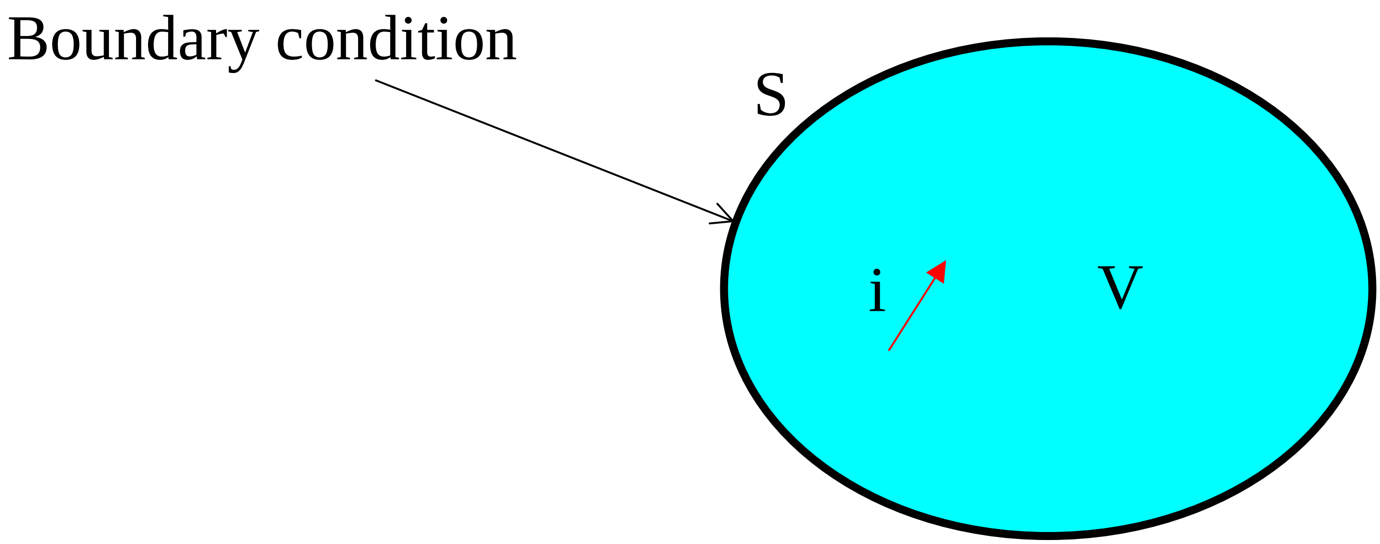
and

$$\nabla^2\phi + k^2\phi = -q$$

straightforward

$$\mathbf{E}, \mathbf{H}$$

Boundary Condition



微分方程式＋境界条件で解が一意に決定！

D.E. + B.C.

Example

D.E. (Differential Equation)

$$\frac{d^2}{dx^2} f(x) = x$$

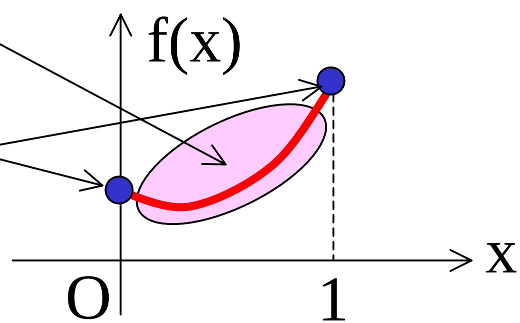
$$f(x) = \frac{x^3}{6} + C_1 x + C_2$$

D.E. +B.C. (Boundary Condition)

$$\frac{d^2}{dx^2} f(x) = x$$

$$f(0) = 0$$

$$f(1) = 1$$



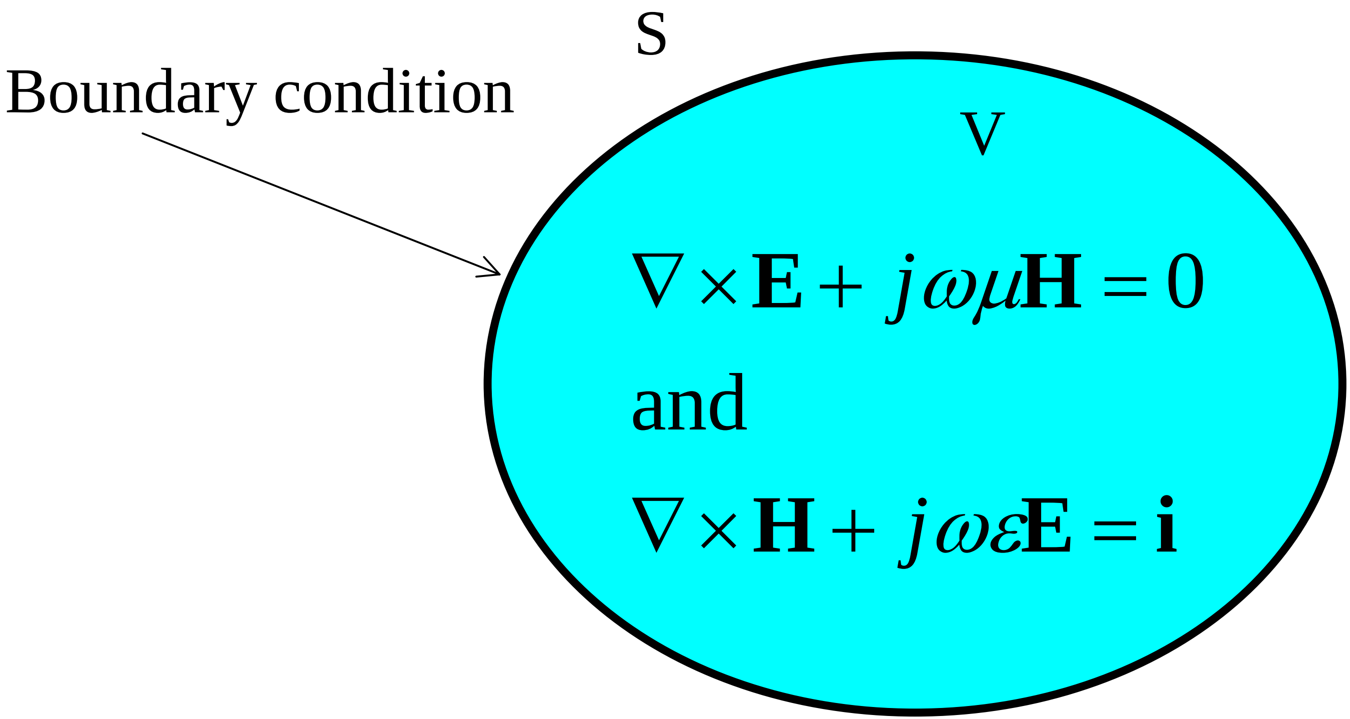
$$f(0) = C_2 = 0$$

$$f(1) = \frac{1}{6} + C_1 = 1, \quad C_1 = \frac{5}{6}$$

$$f(x) = \frac{x}{6}\left(x^2 + 5\right)$$

2階編微分方程式→ 境界値問題 (放物型、楕円型、双曲型)

Maxwell’s Eq. + B.C.



$$\nabla \times \mathbf{E} + j\omega\mu\mathbf{H} = 0$$

and

$$\nabla \times \mathbf{H} + j\omega\varepsilon\mathbf{E} = \mathbf{i}$$

マクスウェルの方程式＋境界条件で解が一意に決定

ある境界条件の下で微分方程式を解きたい！

Numerical Methods

① モーメント法 (Moment Method, Method of Moments, MoM)

- 境界のみに未知数を配置

Variational method,
EMF(Electro Motive Force) method,
ICT(Improved Circuit Theory)
etc.

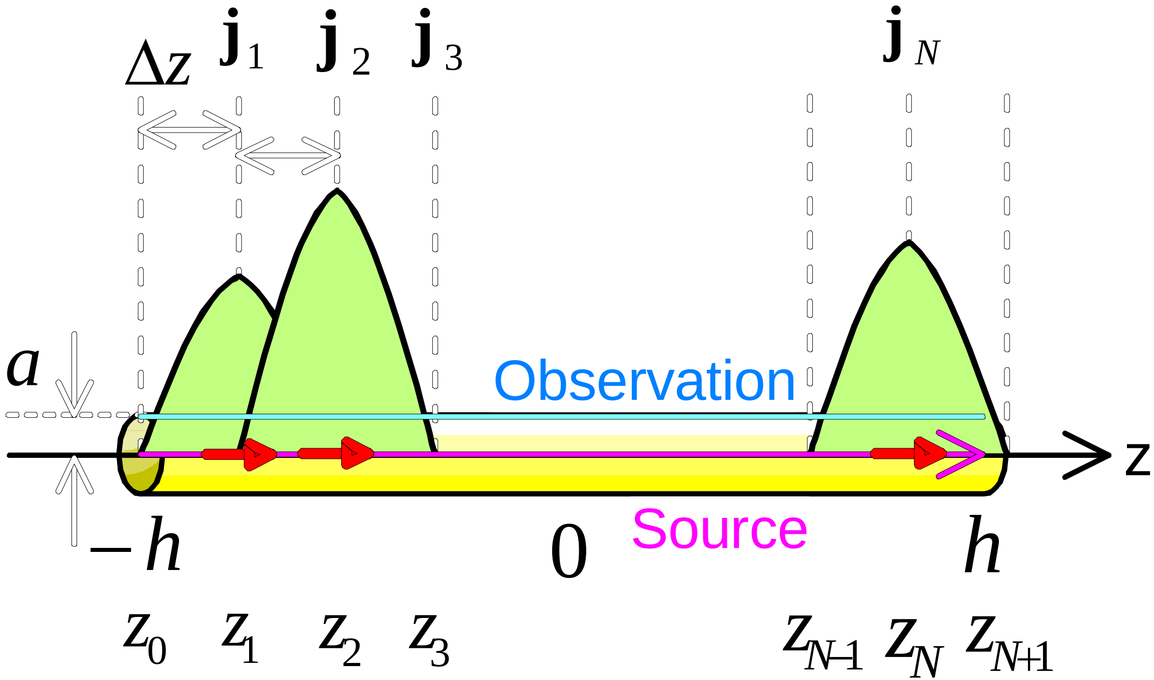
② 有限要素法 (Finite Element Method, FEM)

- 空間を細かく分割し、空間全体に未知数を配置
- 微分方程式を直接解くのでなく、汎関数の極値問題に置き換えて解く。

③ 時間領域差分法 (FDTD 法, Finite Difference Time Domain method)

- マクスウェルの方程式を差分化して電磁界の伝播をシミュレートする

モーメント法
Method of Moments (MoM)
Moment Method



History

E. Hallen,
K. K. Mei,
C. T. Tai

1967 (Harrington)

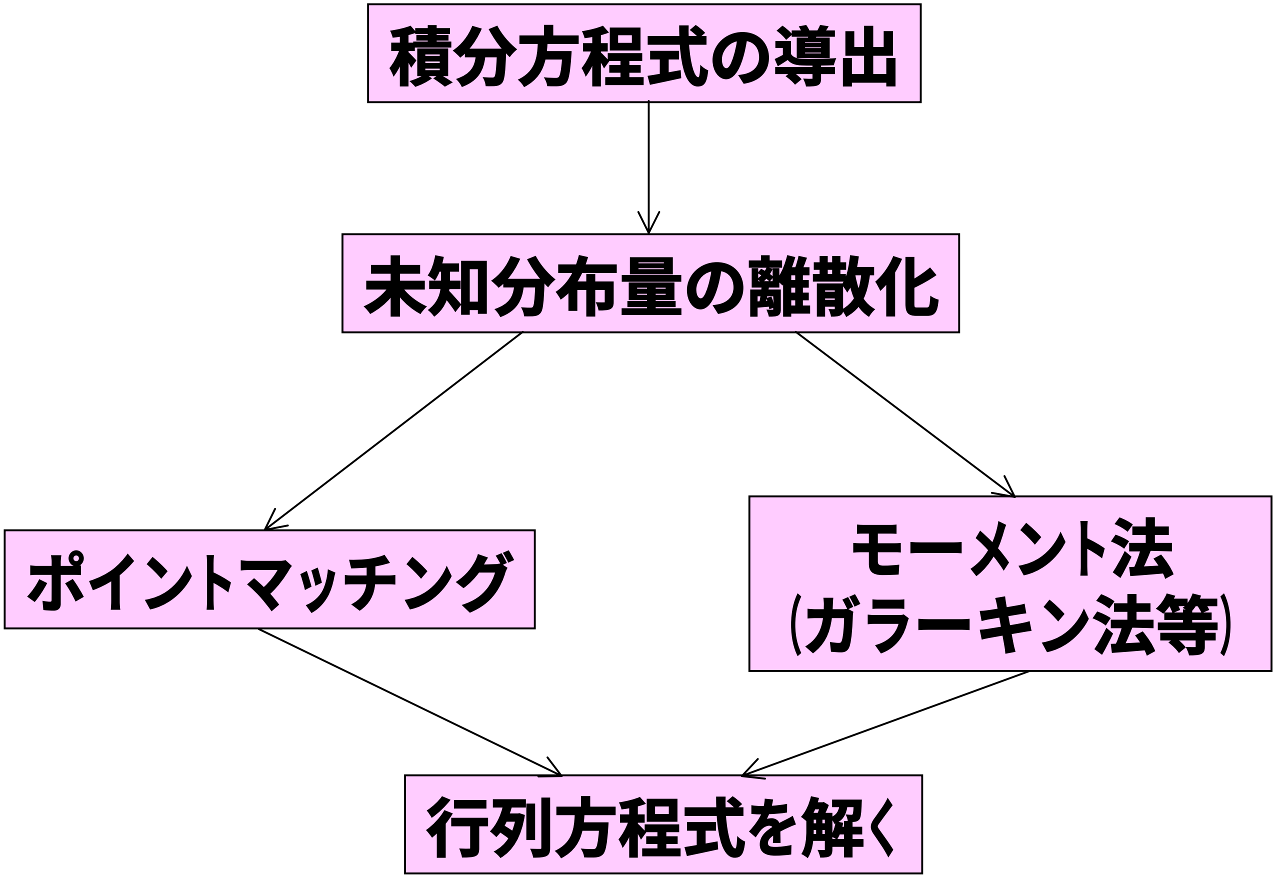
R. F. Harrington, “Matrix Methods for Field Problems”,
Proc. IEEE, vol. 55, pp.136-149, February 1967

R. F. Harrington, and J. R. Mautz, “Straight Wires With Arbitrary Excitation and Loading”,
IEEE Trans. Antenna and Propagat., vol. AP-15, no. 4, pp. 502-514, July 1965

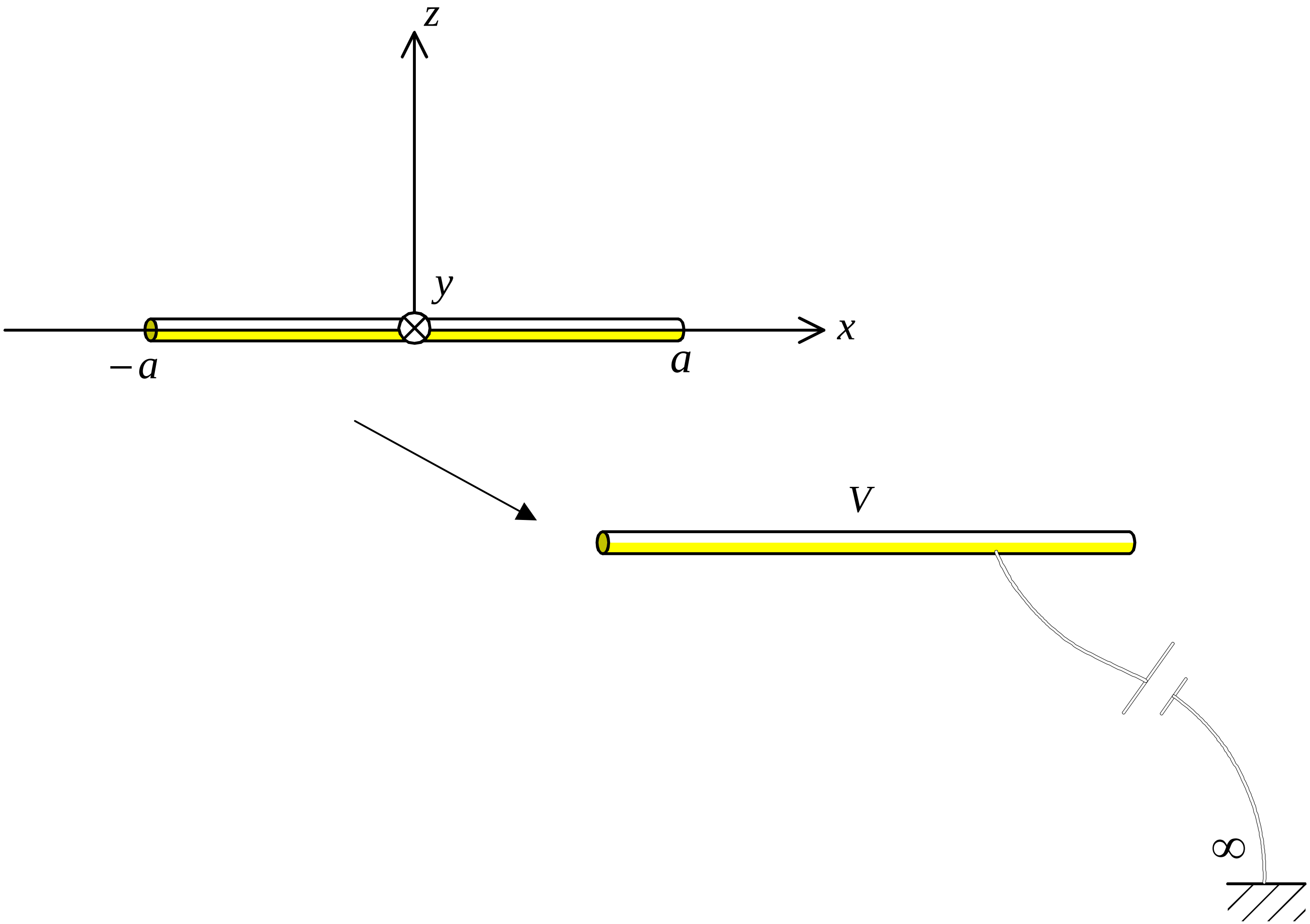
1968

R. F. Harrington, “Field Computation by Moment Methods”,
IEEE Press, New York, 1993

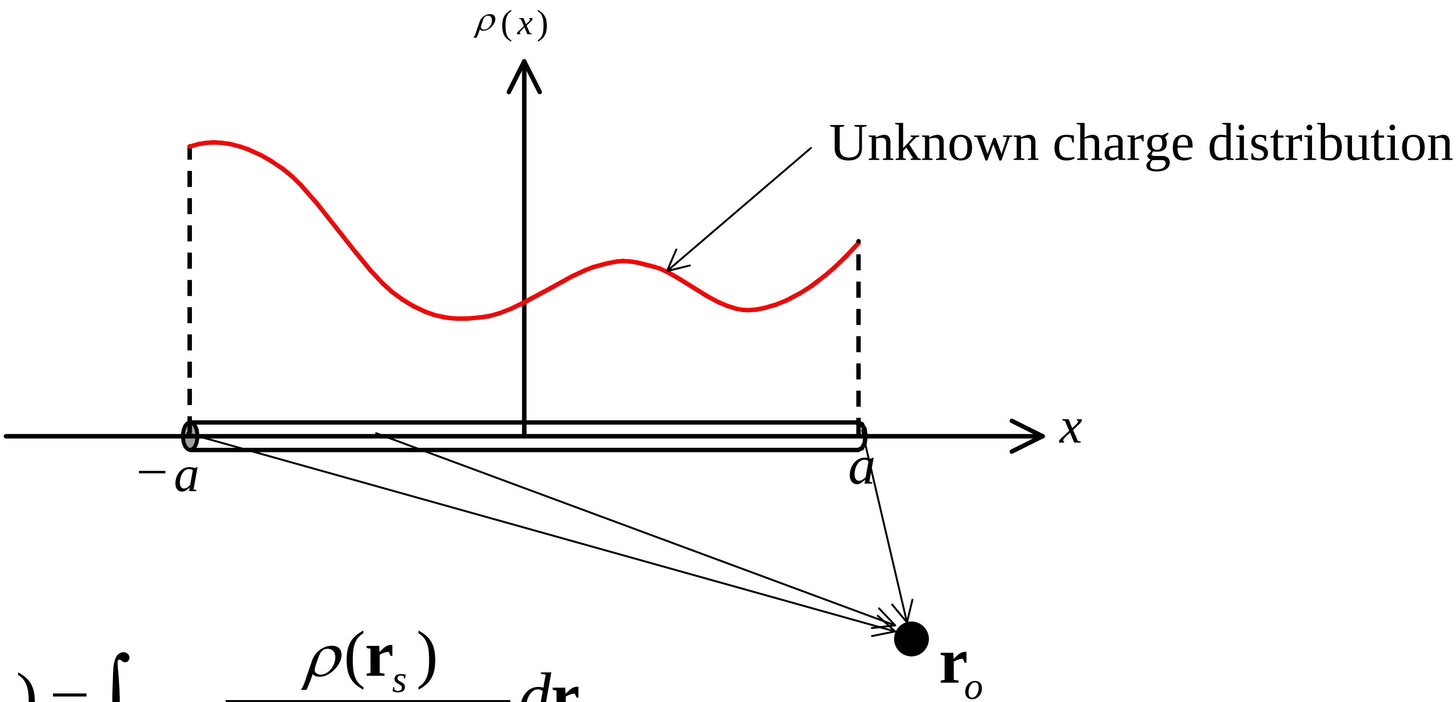
Method of Moments



Example: Charged Wire



Integral Equation

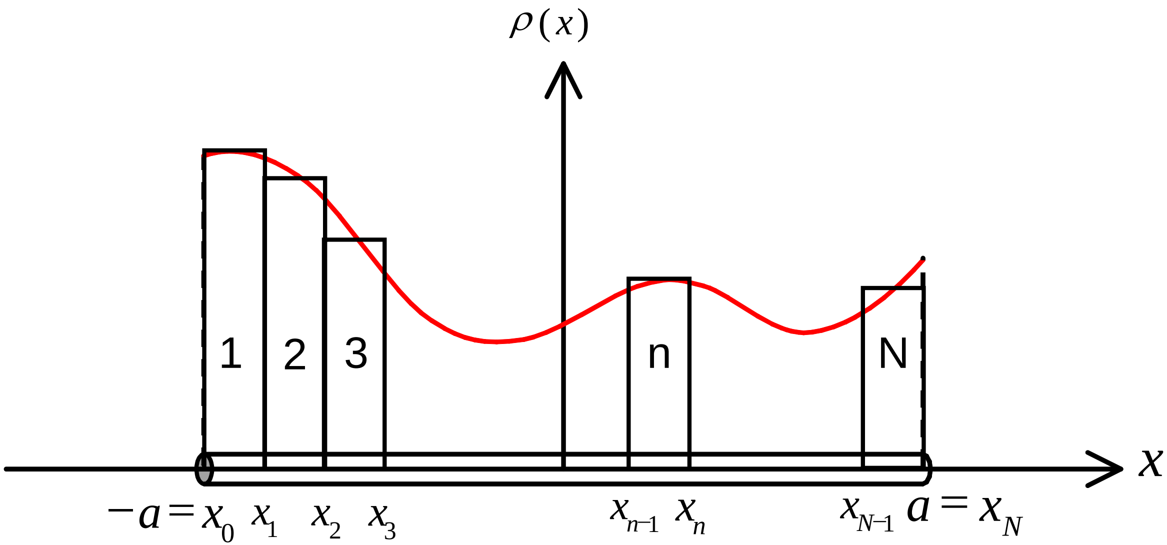


$$\phi(\mathbf{r}_o) = \int_{\text{on bar}} \frac{\rho(\mathbf{r}_s)}{4\pi\varepsilon|\mathbf{r}_o - \mathbf{r}_s|} d\mathbf{r}_s$$

$$\int_{\text{on bar}} \frac{\rho(\mathbf{r}_s)}{4\pi\varepsilon|\mathbf{r}_o - \mathbf{r}_s|} d\mathbf{r}_s = V$$

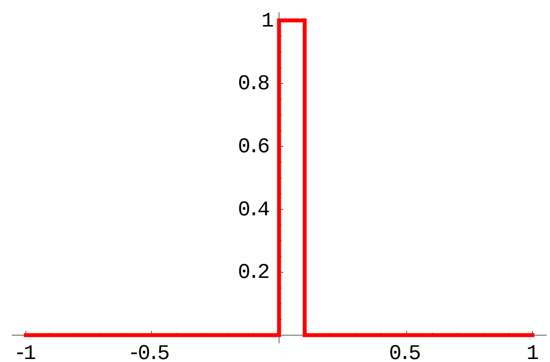
$\mathbf{r}_o$ は導体棒上

Expansion



$$\rho(x) = \sum_{n=1}^{N} a_n f_n(x)$$

$$f_n(x) = \begin{cases} 1 & (x_{n-1} < x < x_n) \\ 0 & (\text{otherwise}) \end{cases}$$

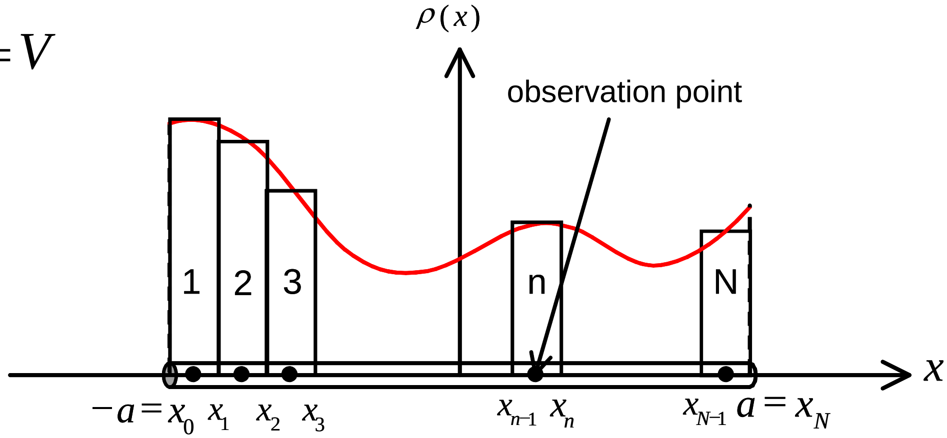


Point Matching

$$\int_{\text{on bar}} \sum_{n=1}^{N} a_n f_n(x) \frac{1}{4\pi\varepsilon|\mathbf{r}_o - \mathbf{r}_s|} d\mathbf{r}_s = V$$

$$\sum_{n=1}^{N} a_n \int_{\text{on bar}} \frac{f_n(x_s)}{4\pi\varepsilon|\mathbf{r}_o - \mathbf{r}_s|} d\mathbf{r}_s = V$$

$$\sum_{n=1}^{N} a_n \int_{x_s = x_{n-1}}^{x_n} \frac{f_n(x_s)}{4\pi\varepsilon|x_o - x_s|} dx_s = V$$



$$x_o{}^n = \frac{x_{n-1} + x_n}{2}$$

$$\sum_{n=1}^{N} a_n \int_{x_s = x_{n-1}}^{x_n} \frac{1}{4\pi\varepsilon|x_o{}^1 - x_s|} dx_s = V$$

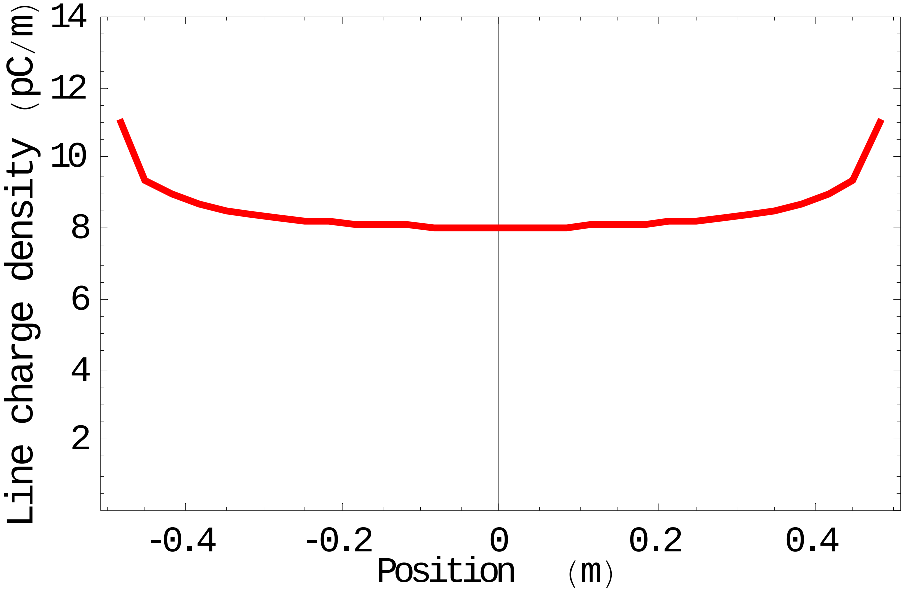
$$\vdots$$

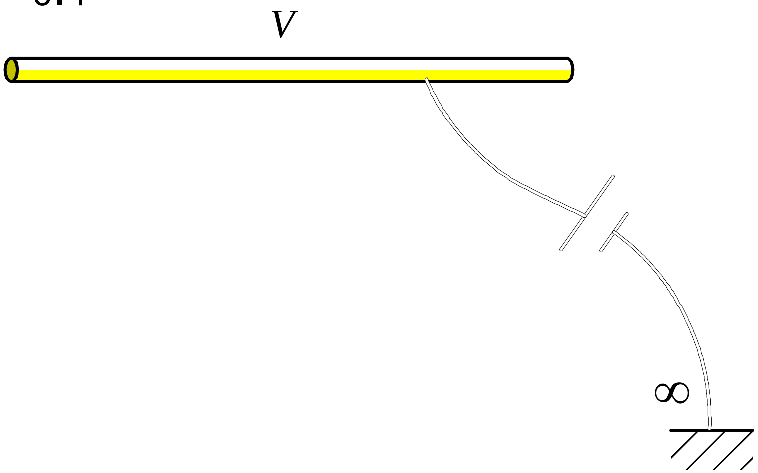
$$\sum_{n=1}^{N} a_n \int_{x_s = x_{n-1}}^{x_n} \frac{1}{4\pi\varepsilon|x_o{}^N - x_s|} dx_s = V$$

$$\begin{bmatrix} z_{11} & \cdots & z_{1N} \\ \vdots & \ddots & \vdots \\ z_{N1} & \cdots & z_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} V \\ \vdots \\ V \end{bmatrix}$$

$$z_{mn} = \int_{x_s = x_{n-1}}^{x_n} \frac{1}{4\pi\varepsilon|x_o{}^m - x_s|} dx_s$$

Example





$$V = 1V, a = 1m$$

$$0.001m$$

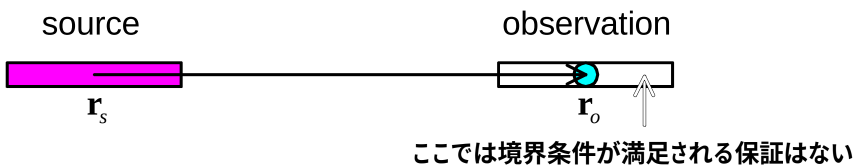
Method of Moments (MoM)

$$\int_{\text{on bar}} \frac{\rho(\mathbf{r}_s)}{4\pi\varepsilon|\mathbf{r}_o - \mathbf{r}_s|} d\mathbf{r}_s = V$$

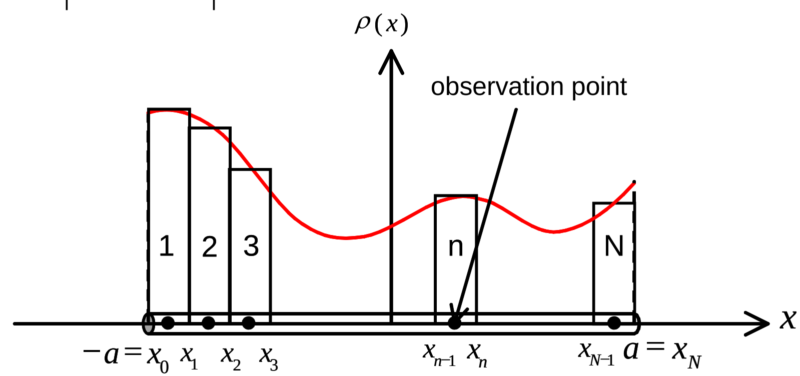
$$\sum_{n=1}^{N} a_n \int_{x_s = x_{n-1}}^{x_n} \frac{1}{4\pi\varepsilon|x_o{}^1 - x_s|} dx_s = V$$

$$\vdots$$

$$\sum_{n=1}^{N} a_n \int_{x_s = x_{n-1}}^{x_n} \frac{1}{4\pi\varepsilon|x_o{}^N - x_s|} dx_s = V$$



(a) **重み付けしない場合**



(b) **重み付けする場合**

観測するときに工夫する

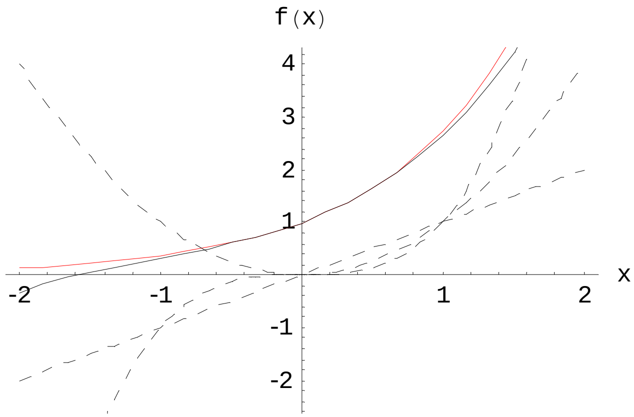
- **全体に境界条件を適用する（重み付け）**

$$\sum_{n=1}^{N} a_n \int_{x_o = x_0}^{x_1} w_1(x_o) \int_{x_s = x_{n-1}}^{x_n} \frac{f_n(x_s)}{4\pi\varepsilon|x_o - x_s|} dx_s dx_o = V$$

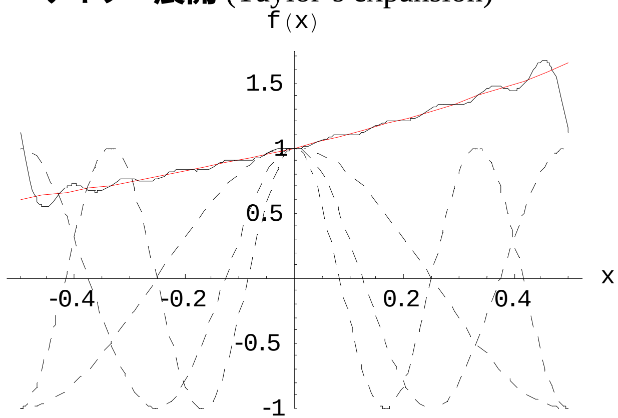
$$\vdots$$

$$\sum_{n=1}^{N} a_n \int_{x_o = x_{N-1}}^{x_N} w_N(x_o) \int_{x_s = x_{n-1}}^{x_n} \frac{f_n(x_s)}{4\pi\varepsilon|x_o - x_s|} dx_s dx_o = V$$

Basis functions

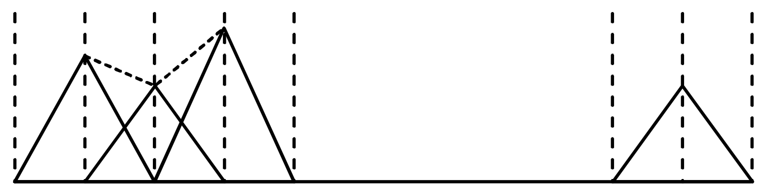


テイラー展開 (Taylor's expansion)



フーリエ級数展開 (Fourier series expansion)

(a) パルス関数

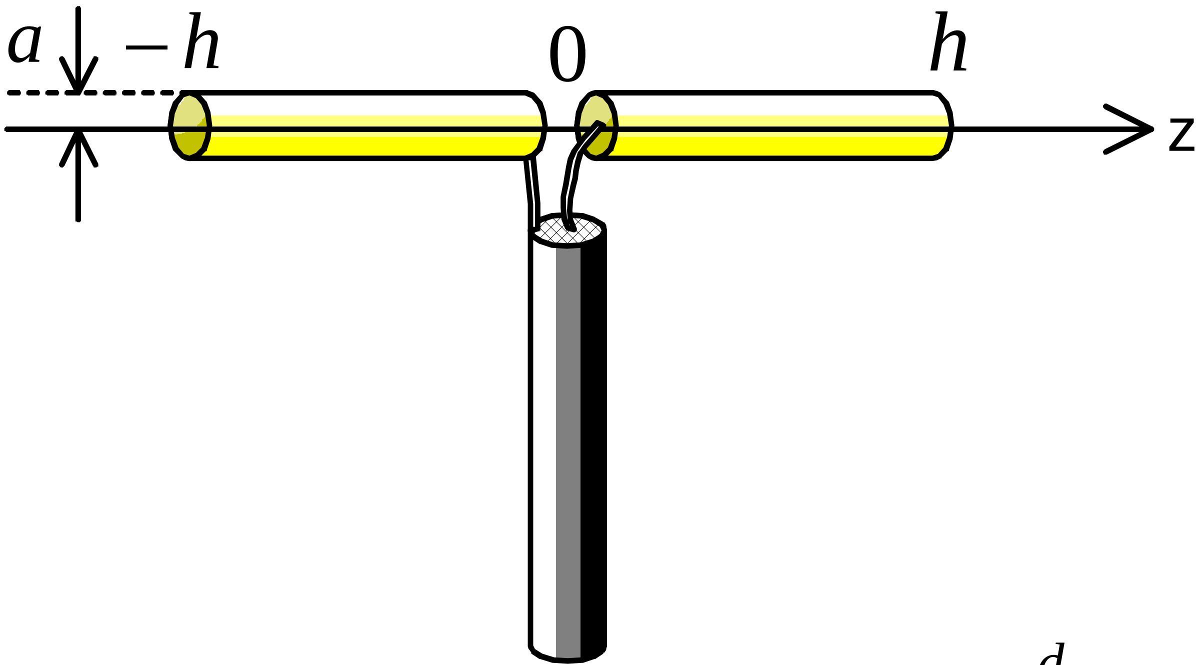


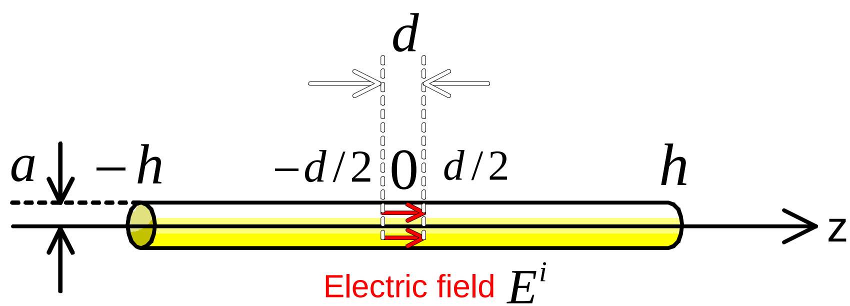
(b) ルーフトップ関数

全域基底関数 (Entire domain basis functions)

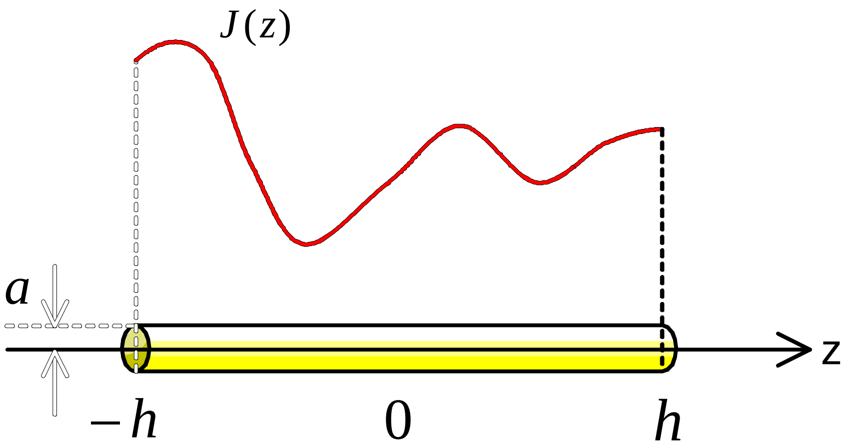
局所基底関数 (local basis functions)

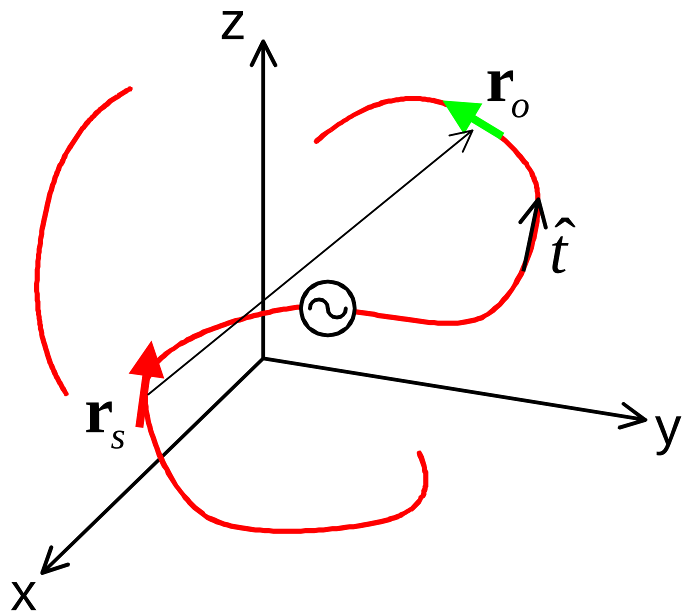
Example: Dipole Antenna





Integral Equation





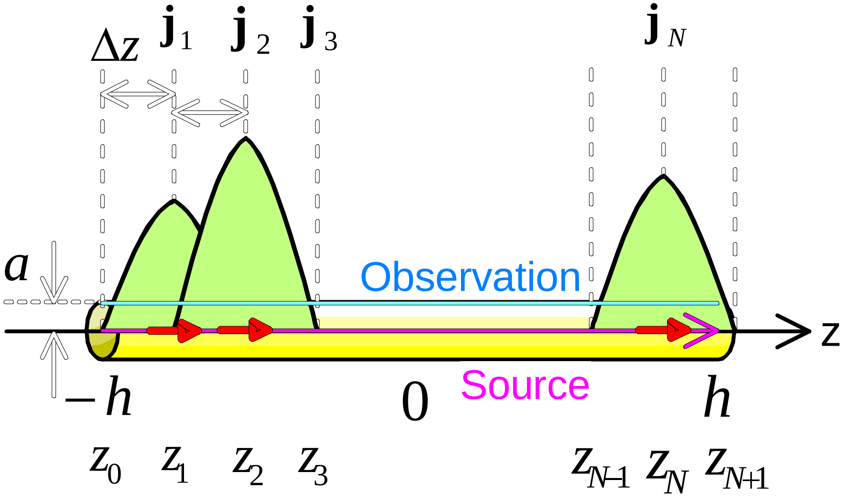
グリーン関数

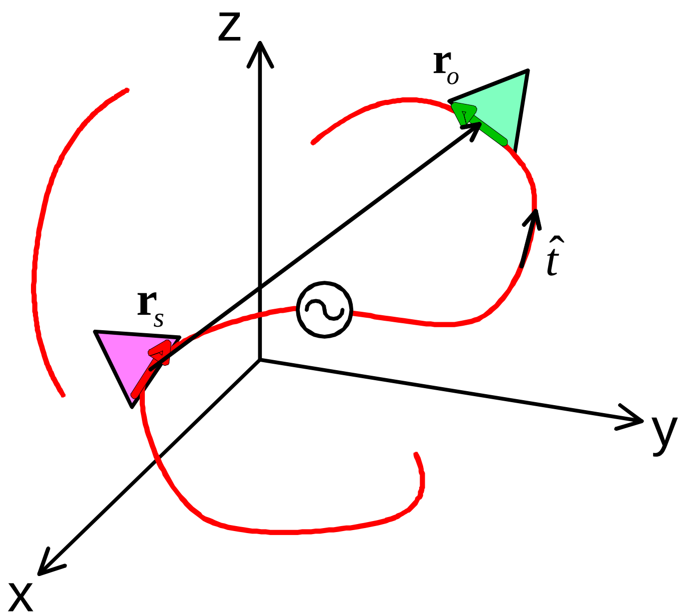
$$\mathbf{E}(\mathbf{r}_o) = \int_{wire} \overline{\mathbf{G}}_{ee}(\mathbf{r}_o;\mathbf{r}_s)\cdot\mathbf{J}(\mathbf{r}_s)d\mathbf{r}_s$$

$$\hat{t}(\mathbf{r}_o)\cdot\left\{\mathbf{E}(\mathbf{r}_o)+\mathbf{E}^i(\mathbf{r}_o)\right\}=0$$ ($\mathbf{r}_o$ is on the wire)

$$\hat{t}(\mathbf{r}_o)\cdot\int_{wire}\overline{\mathbf{G}}_{ee}(\mathbf{r}_o;\mathbf{r}_s)\cdot\mathbf{J}(\mathbf{r}_s)d\mathbf{r}_s = -\hat{t}(\mathbf{r}_o)\cdot\mathbf{E}^i(\mathbf{r}_o)$$

Expansion

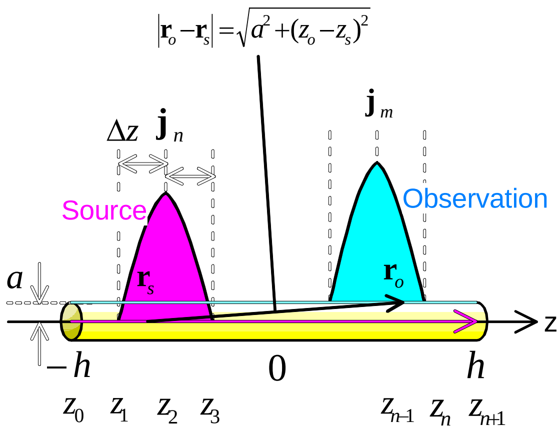


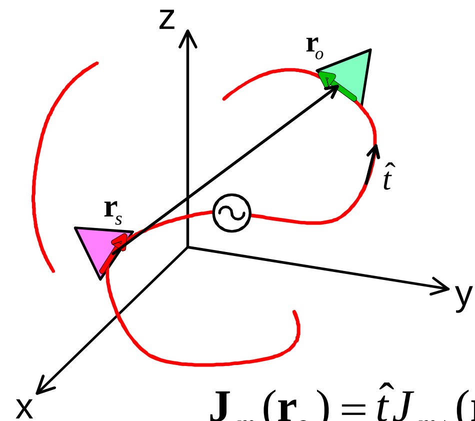


$$\mathbf{J}(\mathbf{r}) = \sum_{n=1}^{N} a_n \mathbf{J}_n(\mathbf{r})$$

$$\hat{t}(\mathbf{r}_o) \cdot \sum_{n=1}^{N} a_n \int_{\Gamma_n} \overline{\mathbf{G}}_{ee}(\mathbf{r}_o;\mathbf{r}_s) \cdot \mathbf{J}_n(\mathbf{r}_s) d\mathbf{r}_s = -\hat{t}(\mathbf{r}_o) \cdot \mathbf{E}^i(\mathbf{r}_o)$$

MoM (Weighting, Averaging)





$$\mathbf{J}_m(\mathbf{r}_o) = \hat{t} J_{mt}(\mathbf{r}_o) = \hat{t}\left\{\hat{t} \cdot \mathbf{J}_m(\mathbf{r}_o)\right\}$$

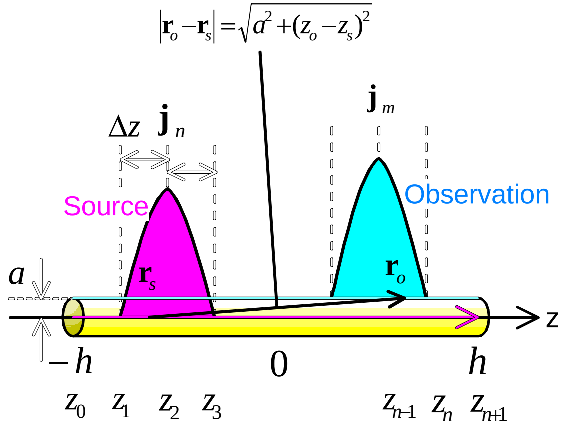
$$\int_{\Gamma_m} \left\{\hat{t}(\mathbf{r}_o) \cdot \mathbf{J}_m(\mathbf{r}_o)\right\}\left[\hat{t}(\mathbf{r}_o) \cdot \sum_{n=1}^{N} a_n \int_{\Gamma_n} \overline{\mathbf{G}}_{ee}(\mathbf{r}_o;\mathbf{r}_s) \cdot \mathbf{J}_n(\mathbf{r}_s) d\mathbf{r}_s\right] d\mathbf{r}_o$$

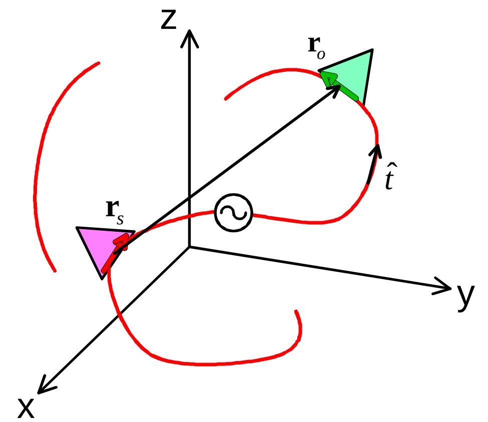
$$= -\int_{\Gamma_m} \left\{\hat{t}(\mathbf{r}_o) \cdot \mathbf{J}_m(\mathbf{r}_o)\right\}\left\{\hat{t}(\mathbf{r}_o) \cdot \mathbf{E}^i(\mathbf{r}_o)\right\} d\mathbf{r}_o$$

$$\int_{\Gamma_m} \mathbf{J}_m(\mathbf{r}_o) \cdot \left[\sum_{n=1}^{N} a_n \int_{\Gamma_n} \overline{\mathbf{G}}_{ee}(\mathbf{r}_o;\mathbf{r}_s) \cdot \mathbf{J}_n(\mathbf{r}_s) d\mathbf{r}_s\right] d\mathbf{r}_o = -\int_{\Gamma_m} \mathbf{J}_m(\mathbf{r}_o) \cdot \mathbf{E}^i(\mathbf{r}_o) d\mathbf{r}_o$$

$$\sum_{n=1}^{N} a_n \int_{\Gamma_m} \mathbf{J}_m(\mathbf{r}_o) \cdot \int_{\Gamma_n} \overline{\mathbf{G}}_{ee}(\mathbf{r}_o;\mathbf{r}_s) \cdot \mathbf{J}_n(\mathbf{r}_s) d\mathbf{r}_s d\mathbf{r}_o = -\int_{\Gamma_m} \mathbf{J}_m(\mathbf{r}_o) \cdot \mathbf{E}^i(\mathbf{r}_o) d\mathbf{r}_o$$

Matrix Form





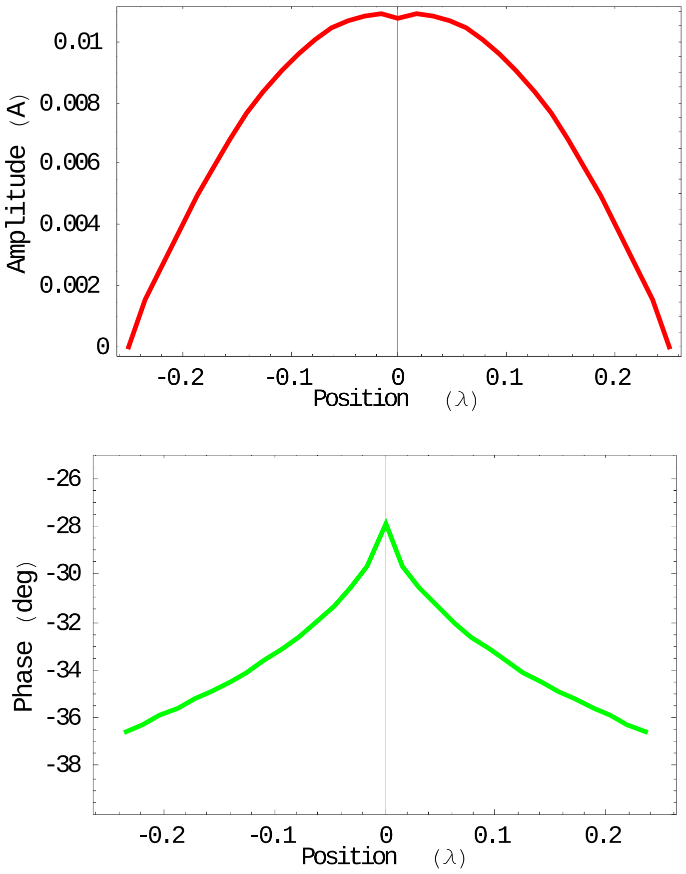
$$
\begin{bmatrix} Z_{11} & \cdots & Z_{1N} \\ \vdots & \ddots & \vdots \\ Z_{N1} & \cdots & Z_{NN} \end{bmatrix}
\begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}
=
\begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix}
$$

観測点を基底関数より多くしたら？？？
条件過剰の方程式を解く問題となる。
（ムーア・ペンローズの一般逆作用素、
QR**分解**, SVD?）

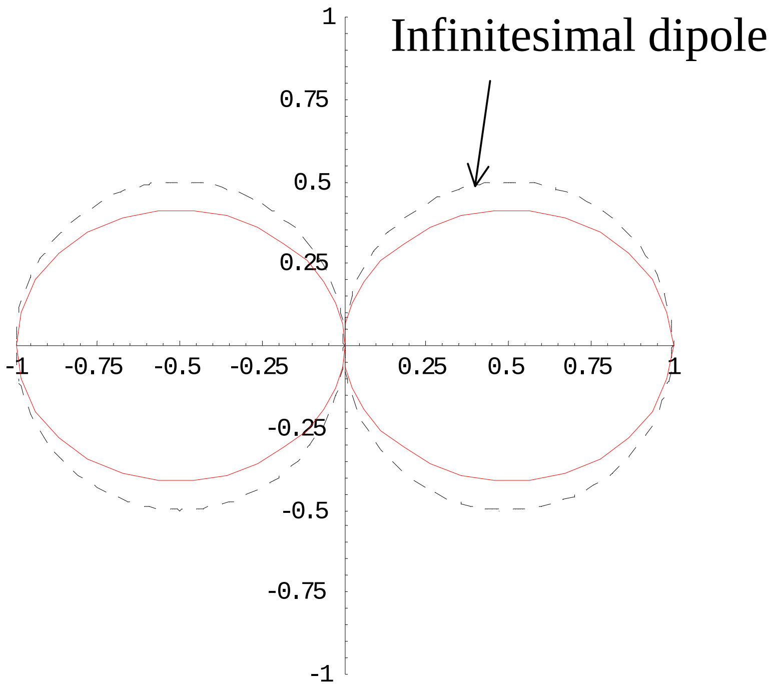
$$
Z_{mn} = \int_{\Gamma_m} \mathbf{J}_m(\mathbf{r}_o) \cdot \underbrace{\int_{\Gamma_n} \overline{\mathbf{G}}_{ee}(\mathbf{r}_o;\mathbf{r}_s) \cdot \mathbf{J}_n(\mathbf{r}_s) d\mathbf{r}_s}_{\mathbf{E}_n(\mathbf{r}_o)} d\mathbf{r}_o = \left\langle \mathbf{J}_m \middle| \overline{\mathbf{G}}_{ee} \middle| \mathbf{J}_n \right\rangle
$$

$$
V_m = -\int_{\Gamma_m} \mathbf{J}_m(\mathbf{r}_o) \cdot \mathbf{E}^i(\mathbf{r}_o) d\mathbf{r}_o = -\left\langle \mathbf{J}_m \middle| \mathbf{E}^i \right\rangle
$$

Half-wavelength Dipole



電流分布



遠方界指向性

Method of Moments

$$\int_a^b K(x,y) f(x) dx = g(y)$$

Fredholm**第一種積分方程式** (参考: 寺沢寛一、「数学概論」、岩波書店)

積分方程式の導出

$$f(x) = \sum_{n=1}^{N} a_n f_n(x)$$

未知分布量の離散化

ポイントマッチング

モーメント法 (ガラーキン法等)

$$\sum_{n=1}^{N} a_n \int_{\Gamma_m} w(y) \underline{\int_{\Gamma_n} K(x,y) f(x) dx dy} = \int_{\Gamma_m} w(y) g(y) dy$$

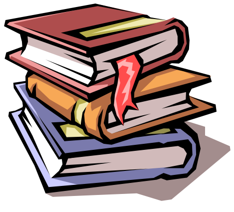
$$\begin{bmatrix} Z_{11} & \cdots & Z_{1N} \\ \vdots & \ddots & \vdots \\ Z_{N1} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix}$$

行列方程式を解く

積分方程式の数値解法！

References

[1] R. F. Harrington, “Field Computation by Moment Methods”, IEEE Press, New York, 1992
[2] C. A. Balanis, “Antenna Theory”, John Wiley & Sons, Inc., 2nd ed., 1997
[3] W. L. Stutzman and G. A. Thiele, “Antenna Theory and Design”, John Wiley & Sons, Inc.



Conclusion

Fine