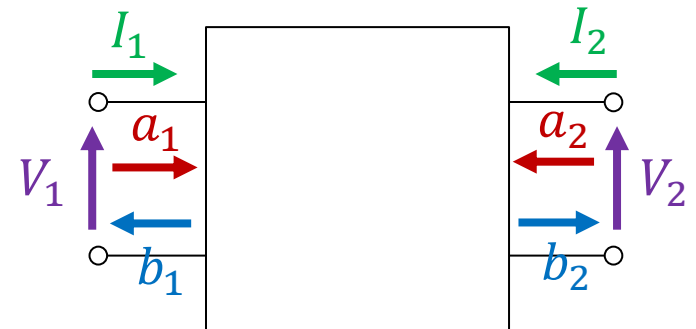
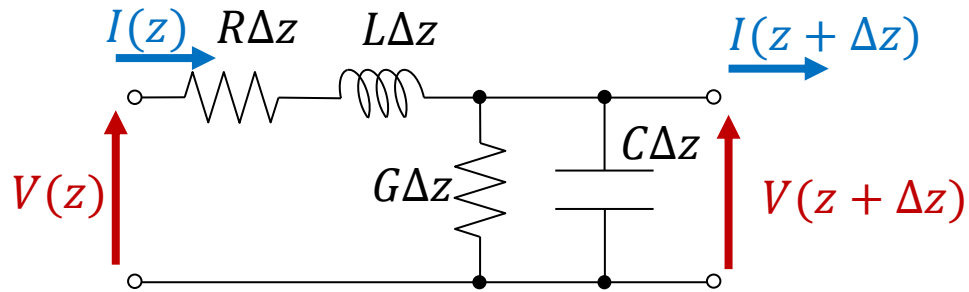


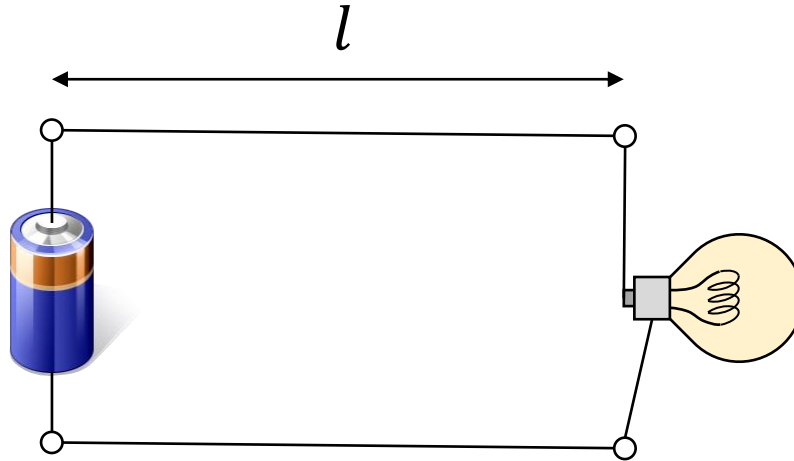
分布定数線路と回路の行列表現



平野拓一

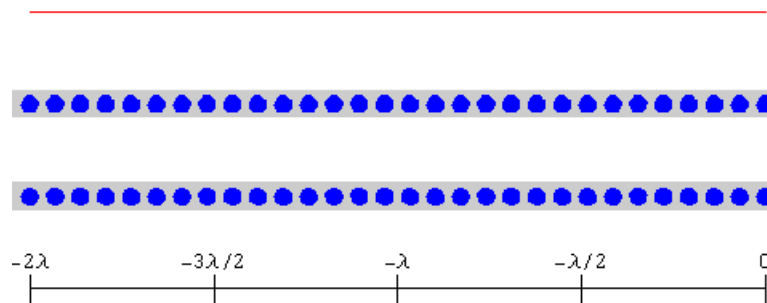
電圧・電流の速度

豆電球が点く時間



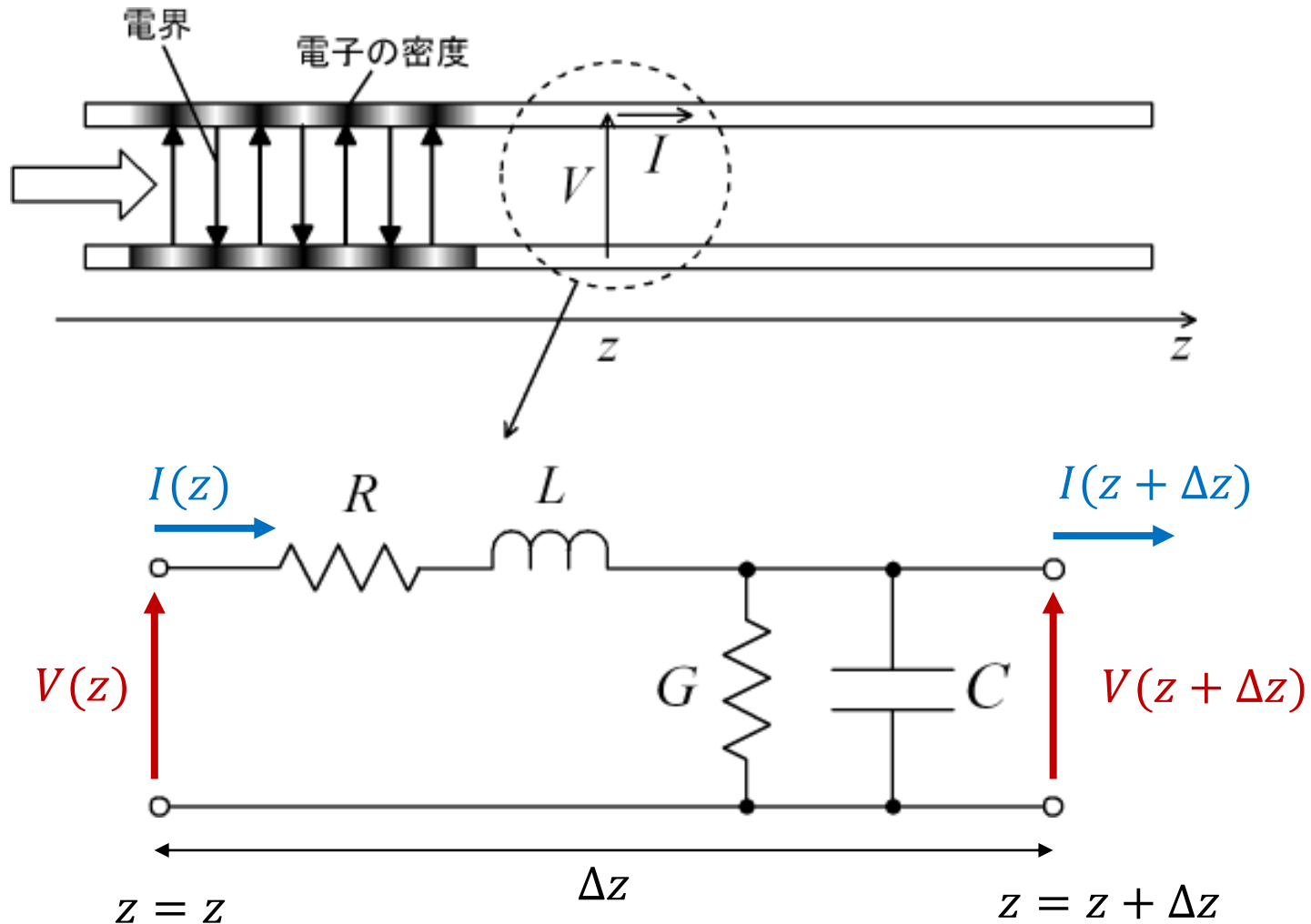
光速: $c \cong 3 \times 10^8$ [m/s]

交流電圧が同じとみなせる距離



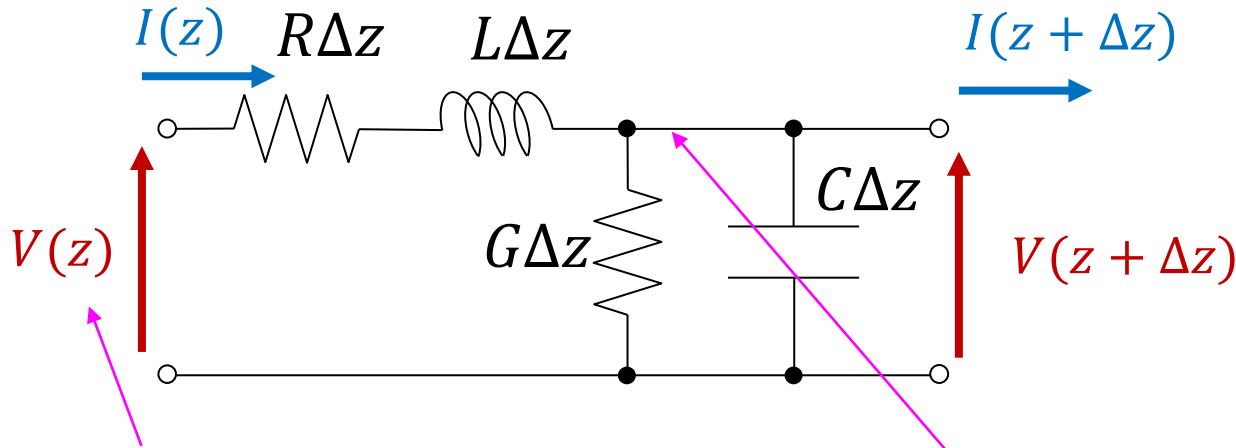
例) $f = 3\text{GHz}$
 $\lambda = c/f = 0.1$ [m]

分布定数線路



1880年代にヘビサイドが構築したモデル。1864年にマクスウェルが電磁波に関する方程式を導出しているので、その類似性から導出したと思われる。電磁波では $L \rightarrow \mu, C \rightarrow \epsilon, G \rightarrow \sigma, V \rightarrow E, I \rightarrow H$ に対応する。

分布定数線路



$$\begin{cases} V(z) = V(z + \Delta z) + (R\Delta z + j\omega L\Delta z)I(z) \\ I(z) - I(z + \Delta z) - (G\Delta z + j\omega C\Delta z)V(z + \Delta z) = 0 \end{cases}$$



$$\begin{cases} -\{V(z + \Delta z) - V(z)\}/\Delta z = (R + j\omega L)I(z) \\ -\frac{\{I(z + \Delta z) - I(z)\}}{\Delta z} = (G + j\omega C)V(z + \Delta z) \end{cases}$$

$$\Delta z \rightarrow 0$$

電信方程式

$$\begin{cases} -\frac{dV}{dz} = (R + j\omega L)I \\ -\frac{dI}{dz} = (G + j\omega C)V \end{cases}$$

分布定数線路

電信方程式

I を消去して連立方程式を1つの変数 V の方程式にする

$$\begin{cases} -\frac{dV}{dz} = \frac{Z}{(R + j\omega L)} I \\ -\frac{dI}{dz} = \frac{Y}{(G + j\omega C)} V \end{cases}$$

$$-\frac{d^2V}{dz^2} = Z \frac{dI}{dz}$$

Maxwell Eq. (1-D)

$$\begin{cases} \frac{\partial E_x}{\partial z} = -j\omega\mu H_y \\ -\frac{dH_y}{\partial z} = j\omega\varepsilon E_x \end{cases}$$

$$\frac{d^2V}{dz^2} - ZYV = 0$$

$$\begin{cases} V = e^{\lambda z} \text{と仮定(常微分方程式[斉次解])} \\ \lambda^2 - ZY = 0 \\ \lambda = \pm\sqrt{ZY} \end{cases}$$

$$\begin{aligned} V &= Ae^{-\sqrt{ZY}z} + Be^{\sqrt{ZY}z} \\ &= Ae^{-\gamma z} + Be^{\gamma z} \quad (\text{伝搬定数 } \gamma = \sqrt{ZY}) \end{aligned}$$

+z方向に進む波 **-z方向に進む波**

$$\begin{aligned} \gamma &= \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)} \\ \gamma &= j\beta = j\omega\sqrt{LC} \quad (\text{無損失 } R = G = 0 \text{ のとき}) \end{aligned}$$

$$\begin{aligned} I &= -\frac{1}{Z} \frac{dV}{dz} \\ &= -\frac{Y}{Z} (-Ae^{-\gamma z} + Be^{\gamma z}) \\ &= \sqrt{\frac{Y}{Z}} (Ae^{-\gamma z} - Be^{\gamma z}) \end{aligned}$$

$1/Z_0$ ($Z_0 = \sqrt{Z/Y}$: 特性インピーダンス, 意味: 電圧と電流の比 $V = Z_0 I$)

$Z_0 = \sqrt{L/C}$ (無損失 $R = G = 0$ のとき)

電圧・電流の表現

$$V = Ae^{-\gamma z} + Be^{\gamma z}$$

$$I = \frac{1}{Z_0} (Ae^{-\gamma z} - Be^{\gamma z})$$

+z方向に進む波 -z方向に進む波

これでもいい

$$V = \sqrt{Z_0} (Ae^{-\gamma z} + Be^{\gamma z})$$

$$I = \frac{1}{\sqrt{Z_0}} (Ae^{-\gamma z} - Be^{\gamma z})$$

+z方向に進む波 -z方向に進む波



電圧・電流を個別に扱うのではなく、進行方向別に分けた電圧・電流の組（または電磁界分布の断面形状）をモードという。

波動関数

$$V = e^{-\gamma z} = e^{-(\alpha + j\beta)z}$$

α : 減衰定数 [Np/m]

β : 位相定数 [rad/m]

なぜこれが波動か？

時間調和振動 (Time-Harmonic Oscillation) と時間波形の関係

http://www.takuichi.net/hobby/edu/em/time_harmonic/time_harmonic.pdf

$$v(t) = \text{Re}[Ve^{j\omega t}] = e^{-\alpha z} \cos(\omega t - \beta z)$$

波動関数の形 $f(z - (\omega / \beta)t)$

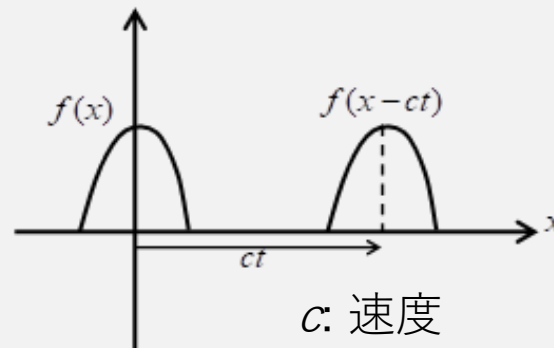
位相速度 $v = \omega / \beta = 1 / \sqrt{LC}$

波動方程式と波動関数

波動方程式 $\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0$



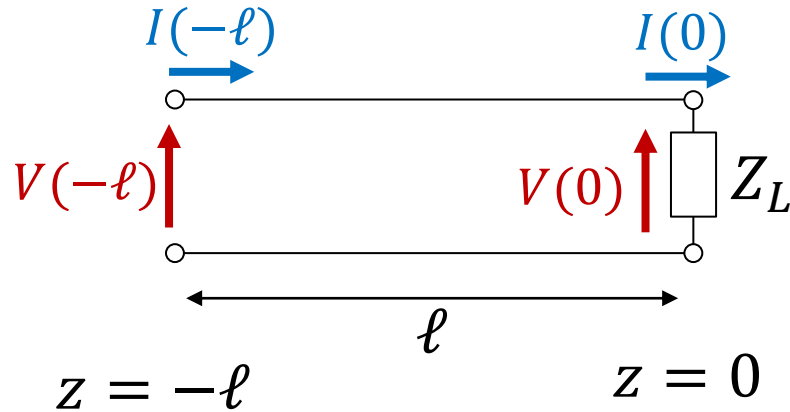
波動関数 $f(x \pm ct)$



$$v = \lambda f = \frac{\lambda}{2\pi} \omega$$

$$\beta = \frac{2\pi}{\lambda}$$

負荷接続分布定数線路の入カインピーダンス



$$V = \sqrt{Z_0}(Ae^{-\gamma z} + Be^{\gamma z})$$

$$I = \frac{1}{\sqrt{Z_0}}(Ae^{-\gamma z} - Be^{\gamma z})$$

+z方向に進む波 -z方向に進む波

at z = 0

$$V(0) = \sqrt{Z_0}(A + B)$$

$$I(0) = \frac{1}{\sqrt{Z_0}}(A - B)$$

$$\frac{V(0)}{I(0)} = Z_0 \frac{A + B}{A - B} = Z_L$$

$$B = \frac{Z_L - Z_0}{Z_L + Z_0} A$$

at z = -ℓ

$$Z_{in} = \frac{V(-\ell)}{I(-\ell)} = Z_0 \frac{Ae^{\gamma \ell} + Be^{-\gamma \ell}}{Ae^{\gamma \ell} - Be^{-\gamma \ell}}$$

$$= Z_0 \frac{e^{\gamma \ell} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma \ell}}{e^{\gamma \ell} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma \ell}}$$

$$= Z_0 \frac{(Z_L + Z_0)e^{\gamma \ell} + (Z_L - Z_0)e^{-\gamma \ell}}{(Z_L + Z_0)e^{\gamma \ell} - (Z_L - Z_0)e^{-\gamma \ell}}$$

分布定数線路の入カインピーダンス

$$\begin{aligned} Z_{in} &= Z_0 \frac{(Z_L + Z_0)e^{\gamma\ell} + (Z_L - Z_0)e^{-\gamma\ell}}{(Z_L + Z_0)e^{\gamma\ell} - (Z_L - Z_0)e^{-\gamma\ell}} \\ &= Z_0 \frac{Z_L(e^{\gamma\ell} + e^{-\gamma\ell}) + Z_0(e^{\gamma\ell} - e^{-\gamma\ell})}{Z_L(e^{\gamma\ell} - e^{-\gamma\ell}) + Z_0(e^{\gamma\ell} + e^{-\gamma\ell})} \\ &= Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)} \end{aligned}$$

無損失線路 $\gamma = j\beta$

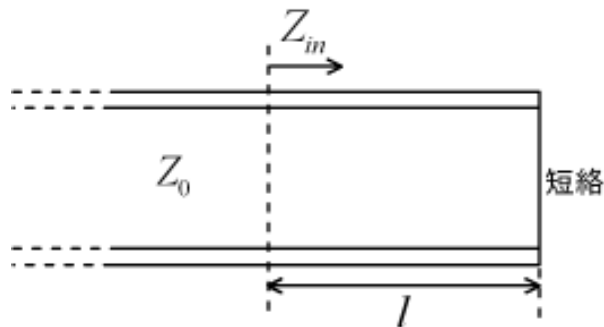
$$Z_{in} = Z_0 \frac{Z_L + j Z_0 \tan(\beta\ell)}{Z_0 + j Z_L \tan(\beta\ell)}$$

$$\begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2} \\ \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}$$

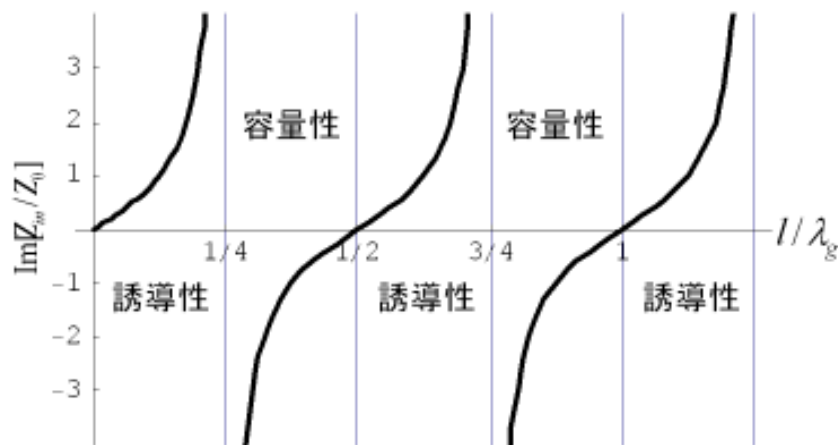
$$\tanh(jx) = j \tan x$$

短絡／開放終端線路

$$Z_{in} = Z_0 \frac{Z_L + j Z_0 \tan(\beta \ell)}{Z_0 + j Z_L \tan(\beta \ell)}$$

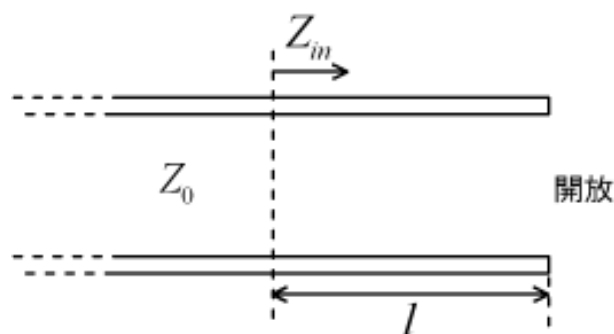


(a)

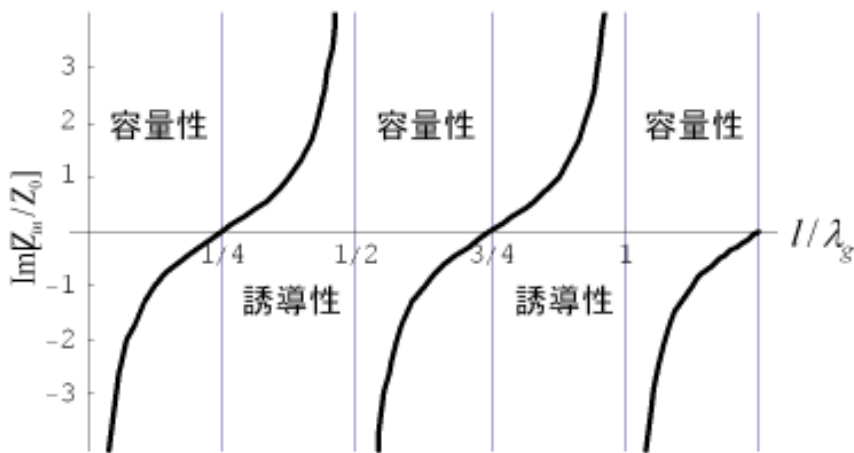


(b)

短絡終端 ($Z_L = 0$) $Z_{in} = j Z_0 \tan(\beta \ell)$



(a)

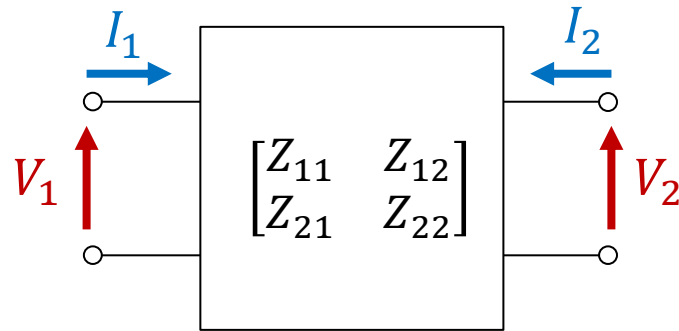


(b)

開放終端 ($Z_L = \infty$) $Z_{in} = -j Z_0 \cot(\beta \ell)$

Z行列

インピーダンス行列

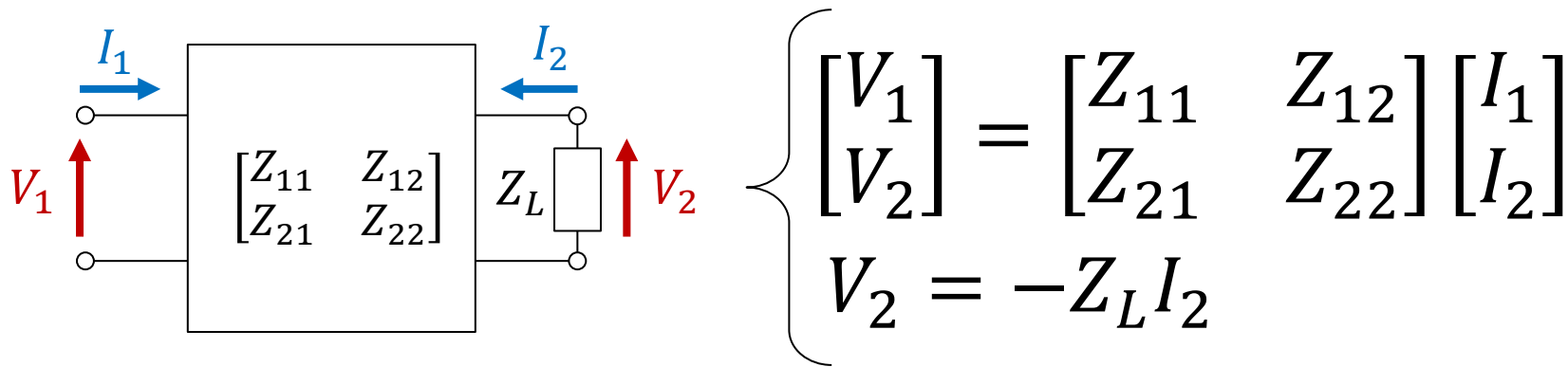


$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$V = ZI$$

二端子対回路、回路構造のブラックボックス化

Z行列の入カインピーダンス

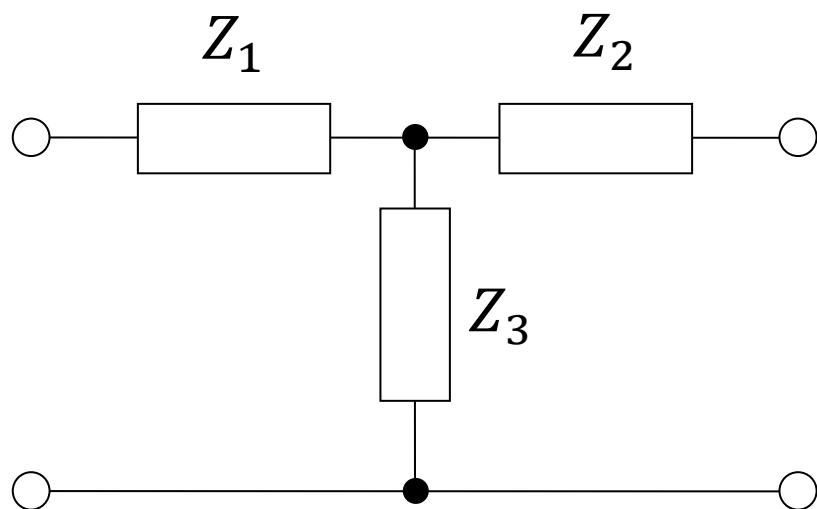


$$\begin{cases} V_1 = Z_{11}I_1 + Z_{12}I_2 \\ V_2 = Z_{21}I_1 + Z_{22}I_2 \end{cases} \implies \begin{aligned} -Z_L I_2 &= Z_{21}I_1 + Z_{22}I_2 \\ -(Z_L - Z_{22})I_2 &= Z_{21}I_1 \\ I_2 &= -\frac{Z_{21}}{Z_L - Z_{22}} I_1 \end{aligned}$$

$$V_1 = \left(Z_{11} - \frac{Z_{12}Z_{21}}{Z_L - Z_{22}} \right) I_1$$

$$Z_{in} = \frac{V_1}{I_1} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_L - Z_{22}}$$

T型回路のZ行列



$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}$$

導出

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

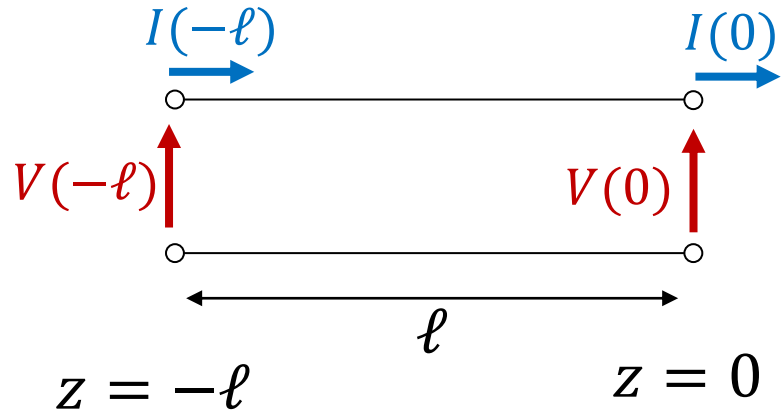
$$Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0} = Z_1 + Z_3$$

$$Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=0} = Z_3$$

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0} = Z_3$$

$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0} = Z_2 + Z_3$$

線路のZ行列



$$V = \sqrt{Z_0}(Ae^{-\gamma z} + Be^{\gamma z})$$

$$I = \frac{1}{\sqrt{Z_0}}(Ae^{-\gamma z} - Be^{\gamma z})$$

+z方向に進む波 -z方向に進む波

at $z = -\ell$

$$V_1 = V(-\ell) = \sqrt{Z_0}(Ae^{\gamma \ell} + Be^{-\gamma \ell})$$

$$I_1 = I(-\ell) = \frac{1}{\sqrt{Z_0}}(Ae^{\gamma \ell} - Be^{-\gamma \ell})$$

at $z = 0$

$$V_2 = V(0) = \sqrt{Z_0}(A + B)$$

$$I_2 = -I(0) = \frac{-1}{\sqrt{Z_0}}(A - B)$$

$$\begin{cases} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \sqrt{Z_0} \begin{bmatrix} e^{\gamma \ell} & e^{-\gamma \ell} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \\ \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\sqrt{Z_0}} \begin{bmatrix} e^{\gamma \ell} & -e^{-\gamma \ell} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \end{cases}$$

線路のZ行列

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \sqrt{Z_0} \begin{bmatrix} e^{\gamma \ell} & e^{-\gamma \ell} \\ 1 & 1 \end{bmatrix} \left(\frac{1}{\sqrt{Z_0}} \begin{bmatrix} e^{\gamma \ell} & -e^{-\gamma \ell} \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Z matrix

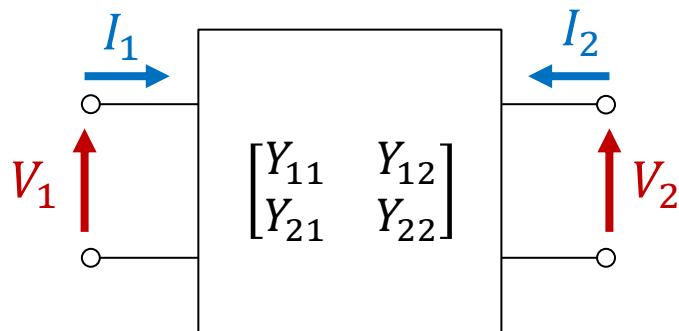
$$Z = \begin{bmatrix} Z_0 \coth(\gamma \ell) & Z_0 \operatorname{csch}(\gamma \ell) \\ Z_0 \operatorname{csch}(\gamma \ell) & Z_0 \coth(\gamma \ell) \end{bmatrix}$$

無損失線路 $\gamma = j\beta$

$$Z = \begin{bmatrix} -j Z_0 \cot(\beta \ell) & -j Z_0 \operatorname{csc}(\beta \ell) \\ -j Z_0 \operatorname{csc}(\beta \ell) & -j Z_0 \cot(\beta \ell) \end{bmatrix}$$

Y行列

アドミタンス行列



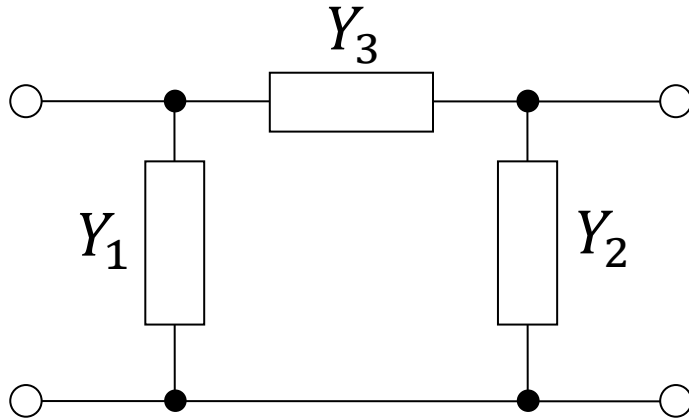
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I = YV$$

$$Y = Z^{-1}$$

$$Z = Y^{-1}$$

π 型回路のY行列



$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_1 + Y_3 & -Y_3 \\ -Y_3 & Y_2 + Y_3 \end{bmatrix}$$

導出

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$Y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0} = Y_1 + Y_3$$

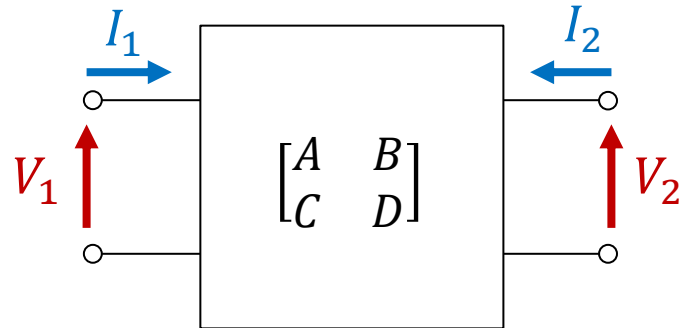
$$Y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1=0} = -Y_3$$

$$Y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2=0} = -Y_3$$

$$Y_{22} = \left. \frac{I_2}{V_2} \right|_{V_1=0} = Y_2 + Y_3$$

F行列

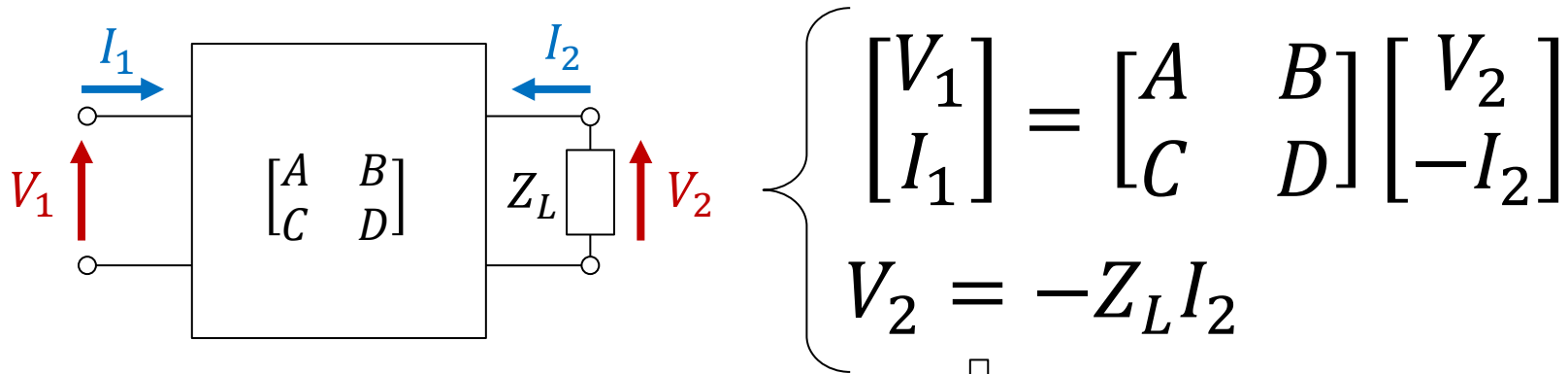
縦続行列, Fundamental Matrix



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

縦続接続された回路網の解析に便利。

F行列の入カインピーダンス



$$\begin{cases} V_1 = AV_2 - BI_2 \\ I_1 = CV_2 - DI_2 \end{cases}$$

$$\begin{cases} V_1 = -(AZ_L + B)I_2 \\ I_1 = -(CZ_L + D)I_2 \end{cases}$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{AZ_L + B}{CZ_L + D}$$

Z行列 \Rightarrow F行列の変換

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

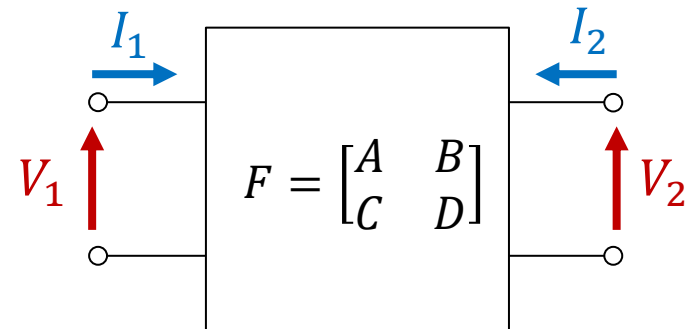
$$\begin{cases} V_1 = Z_{11}I_1 + Z_{12}I_2 \\ V_2 = Z_{21}I_1 + Z_{22}I_2 \end{cases}$$

$$\begin{cases} V_1 - Z_{11}I_1 = -Z_{12}(-I_2) \\ Z_{21}I_1 = V_2 + Z_{22}(-I_2) \end{cases}$$

$$\begin{bmatrix} 1 & -Z_{11} \\ 0 & Z_{21} \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 0 & -Z_{12} \\ 1 & Z_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & -Z_{11} \\ 0 & Z_{21} \end{bmatrix}^{-1} \begin{bmatrix} 0 & -Z_{12} \\ 1 & Z_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$F = \frac{1}{Z_{21}} \begin{bmatrix} Z_{11} & |Z| \\ 1 & Z_{22} \end{bmatrix}$$



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

F行列 \Rightarrow Z行列の変換

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

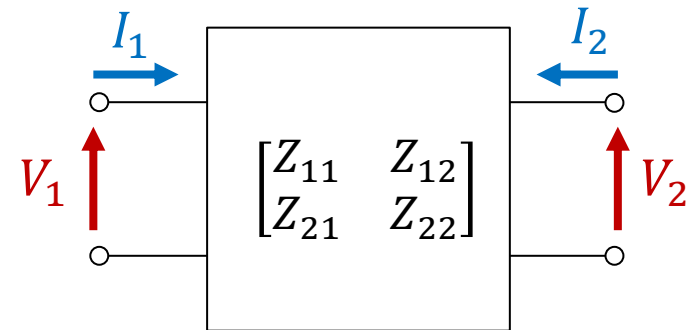
$$\begin{cases} V_1 = AV_2 - BI_2 \\ I_1 = CV_2 - DI_2 \end{cases}$$

$$\begin{cases} V_1 - AV_2 = -BI_2 \\ CV_2 = I_1 + DI_2 \end{cases}$$

$$\begin{bmatrix} 1 & -A \\ 0 & C \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & -B \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & -A \\ 0 & C \end{bmatrix}^{-1} \begin{bmatrix} 0 & -B \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

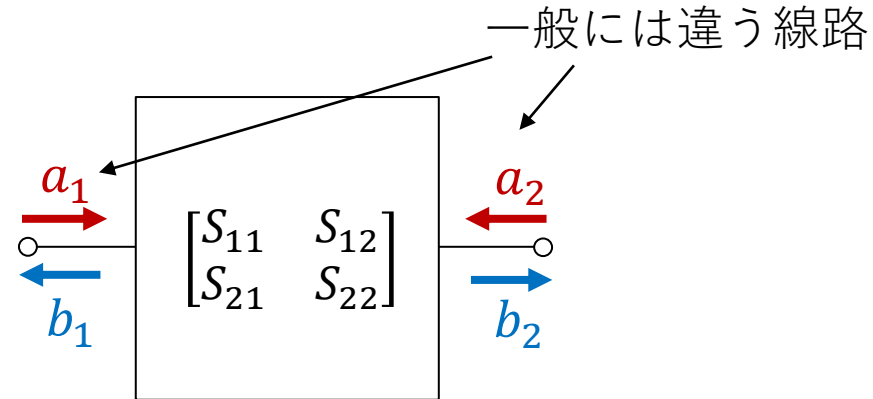
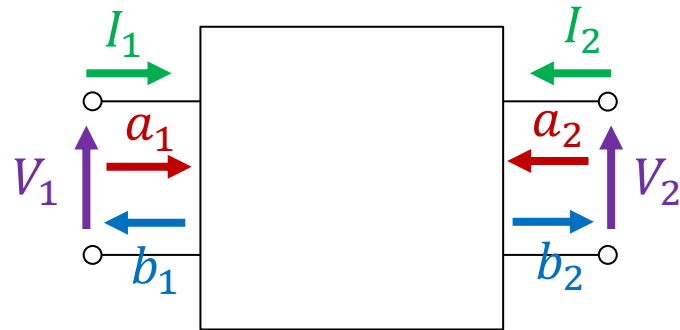
$$Z = \frac{1}{C} \begin{bmatrix} A & |F| \\ 1 & D \end{bmatrix}$$



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

S行列

散乱行列, Scattering Matrix, Sパラメータ



$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

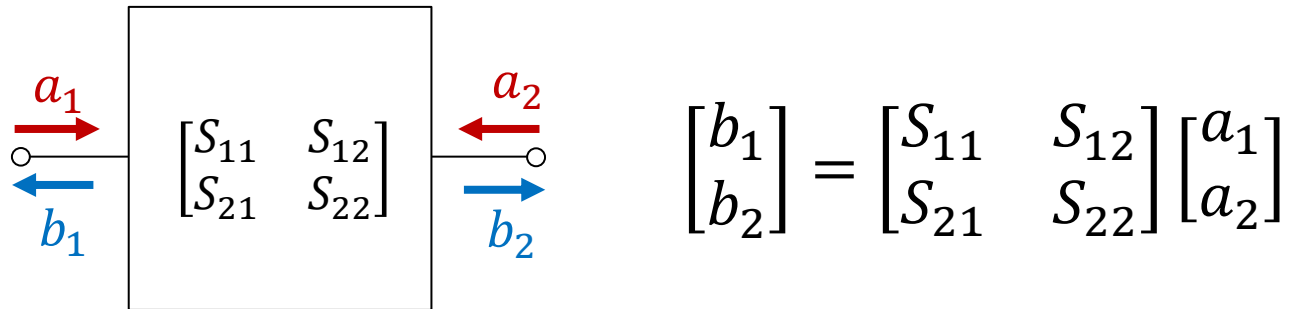
入射波・反射波の重みで表現する

$$V = \sqrt{Z_0} (Ae^{-\gamma z} + Be^{\gamma z})$$

$$I = \frac{1}{\sqrt{Z_0}} (Ae^{-\gamma z} - Be^{\gamma z})$$

+z方向に進む波 -z方向に進む波

反射係数と透過係数



ポート1を励振するとき

$$\text{反射係数 : } S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2=0}$$

$$\text{透過係数 : } S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2=0}$$

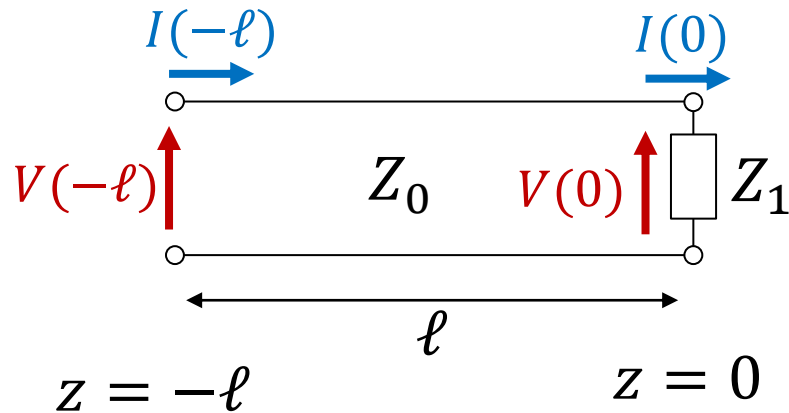
ポート2を励振するとき

$$\text{反射係数 : } S_{22} = \left. \frac{b_2}{a_2} \right|_{a_1=0}$$

$$\text{透過係数 : } S_{12} = \left. \frac{b_1}{a_2} \right|_{a_1=0}$$

つまり、 S_{ij} はポート*j*からポート*i*への散乱係数

線路と負荷 / 線路と線路の反射係数



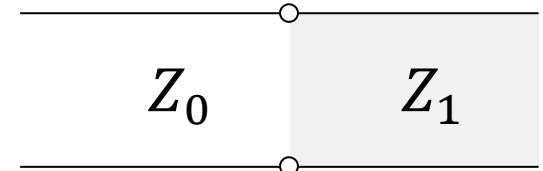
$$V = \sqrt{Z_0}(Ae^{-\gamma z} + Be^{\gamma z})$$

$$I = \frac{1}{\sqrt{Z_0}}(Ae^{-\gamma z} - Be^{\gamma z})$$

+z方向に進む波 -z方向に進む波

反射係数： $S_{11} = \frac{Be^{-\gamma \ell}}{Ae^{\gamma \ell}} = \frac{Z_1 - Z_0}{Z_1 + Z_0} e^{-2\gamma \ell}$

p.8より



$\ell = 0$ (端面)の反射係数

$$S_{11} = \begin{cases} -1 & (Z_1 = 0) \\ 1 & (Z_1 = \infty) \\ 0 & (Z_1 = Z_0) \end{cases}$$

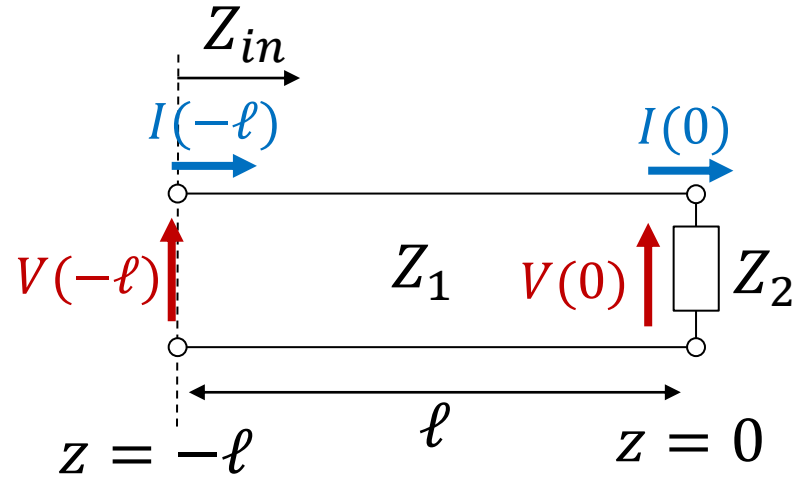
$$S_{11} = \frac{Z_1 - Z_0}{Z_1 + Z_0} \iff Z_1 = Z_0 \frac{1 + S_{11}}{1 - S_{11}}$$

S_{11} と Z_0 がわかれば Z_L (入カインピーダンス Z_{11})が計算できる。

1/4波長整合器

無損失線路 $\gamma = j\beta$

p.9
$$Z_{in} = Z_1 \frac{Z_2 + jZ_1 \tan(\beta\ell)}{Z_1 + jZ_2 \tan(\beta\ell)}$$

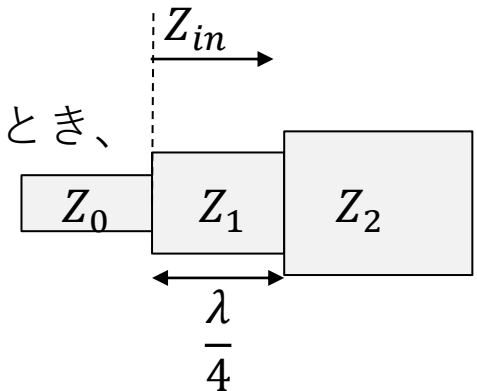


ここで、 $z < -\ell$ の線路の特性インピーダンスが Z_0 のとき、 $Z_{in} = Z_0$ であれば反射はなくなる。その条件を求めると、

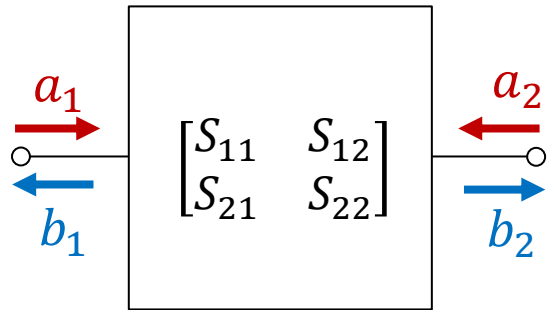
$$Z_0 = Z_1 \frac{Z_2 + jZ_1 \tan(\beta\ell)}{Z_1 + jZ_2 \tan(\beta\ell)}$$

$\tan(\beta\ell) = \infty$ 、つまり $\beta\ell = \pi/2$ ($\ell = \frac{\pi}{2\beta} = \frac{\lambda}{4}$)のとき、

$$Z_{in} = Z_0 = Z_1 \frac{Z_1}{Z_2} \implies Z_1 = \sqrt{Z_0 Z_2}$$



S行列とZ行列の関係

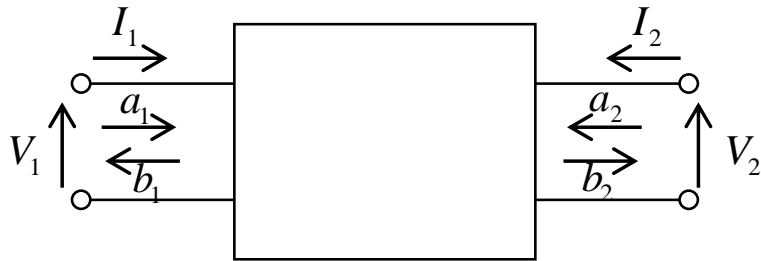


$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

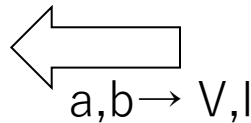
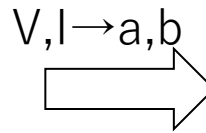
$$V = \sqrt{Z_0} (Ae^{-\gamma z} + Be^{\gamma z})$$

$$I = \frac{1}{\sqrt{Z_0}} (Ae^{-\gamma z} - Be^{\gamma z})$$

+z方向に進む波 -z方向に進む波



$$\begin{cases} V_i = (a_i + b_i)\sqrt{Z_i} \\ I_i = \frac{a_i - b_i}{\sqrt{Z_i}} \end{cases}$$



$$\begin{cases} a_i = \frac{V_i + Z_i I_i}{2\sqrt{Z_i}} \\ b_i = \frac{V_i - Z_i I_i}{2\sqrt{Z_i}} \end{cases}$$

ミリ波帯CMOSオンチップ線路など損失のある線路では特性インピーダンスは複素数になる。その場合には下記参

T. Hirano, "Review and Another Derivation of the Power Wave," Microwave and Optical Technology Letters (MOP), Vol.57, No.1, pp.26-28, DOI: 10.1002/mop.28766, January 2015.

Z行列⇒S行列の変換

$$\begin{cases} a_1 = \frac{V_1 + Z_1 I_1}{2\sqrt{Z_1}} \\ b_1 = \frac{V_1 - Z_1 I_1}{2\sqrt{Z_1}} \end{cases} \quad \begin{cases} a_2 = \frac{V_2 + Z_2 I_2}{2\sqrt{Z_2}} \\ b_2 = \frac{V_2 - Z_2 I_2}{2\sqrt{Z_2}} \end{cases}$$

一般に

$$\begin{cases} a_i = \frac{V_i + Z_i I_i}{2\sqrt{Z_i}} \\ b_i = \frac{V_i - Z_i I_i}{2\sqrt{Z_i}} \end{cases}$$

$$\begin{aligned} V^{-1} &= U \\ U^{-1} &= V \end{aligned}$$

$$\begin{cases} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{2} V \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \frac{1}{2} U \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{2} V \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - \frac{1}{2} U \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \end{cases}$$

$$U = \text{diag}(\sqrt{Z_i}) \quad UV = I \text{ (単位行列)}$$

$$\begin{cases} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \sqrt{Z_1} & 0 \\ 0 & \sqrt{Z_2} \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\ \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{Z_1} & 0 \\ 0 & 1/\sqrt{Z_2} \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \end{cases}$$

$$\begin{cases} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{2} (VZ + U) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{2} (VZ - U) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \end{cases}$$

$$V = \text{diag}(1/\sqrt{Z_i})$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{2} (VZ - U) \left[\frac{1}{2} (VZ + U) \right]^{-1} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

S

$$\begin{aligned} S &= \frac{1}{2} (VZ - U) \left[\frac{1}{2} (VZ + U) \right]^{-1} \\ &= (VZ - U) V V^{-1} (VZ + U)^{-1} \\ &= (VZV - VU) (VZV + UV)^{-1} \\ &= (VZV - I) (VZV + I)^{-1} \end{aligned}$$

Z行列 \Rightarrow S行列の変換(別表現)

$$\begin{aligned} S &= (\hat{Z}VZV - I)(VZV + I)^{-1} \\ &= (\hat{Z} - I)(\hat{Z} + I)^{-1} \end{aligned}$$

$$\left(\begin{array}{l} \text{ここで、} \\ (\hat{Z} - I)(\hat{Z} + I) = \hat{Z}^2 + I = (\hat{Z} + I)(\hat{Z} - I) \\ \text{左および右から}(\hat{Z} + I)^{-1}\text{を掛けると} \\ (\hat{Z} + I)^{-1}(\hat{Z} - I) = (\hat{Z} - I)(\hat{Z} + I)^{-1} \end{array} \right)$$
$$= (\hat{Z} + I)^{-1}(\hat{Z} - I)$$

S行列 \Rightarrow Z行列の変換

$$S = (VZV + I)^{-1} (VZV - I)$$

$$(VZV + I)S = (VZV - I)$$

$$VZVS + S = VZV - I$$

$$VZV - VZVS = I + S$$

$$VZV(I - S) = I + S$$

$$VZV = (I + S)(I - S)^{-1}$$

$$VZV = (I + S)(I - S)^{-1}$$

$$Z = U(I + S)(I - S)^{-1}U$$

I : 単位行列

$$U = \text{diag}(\sqrt{Z_i}) = \begin{bmatrix} \sqrt{Z_1} & 0 \\ 0 & \sqrt{Z_2} \end{bmatrix}$$

S行列 \Rightarrow Z行列の変換(別表現)

$$Z = U(I + S)(I - S)^{-1}U$$

$$\left(\begin{array}{l} U^{-1}ZU^{-1} = (I + S)(I - S)^{-1} \\ \text{右辺はp.25同様に交換可能。} \\ U^{-1}ZU^{-1} = (I - S)^{-1}(I + S) \end{array} \right)$$

$$= U(I - S)^{-1}(I + S)U$$

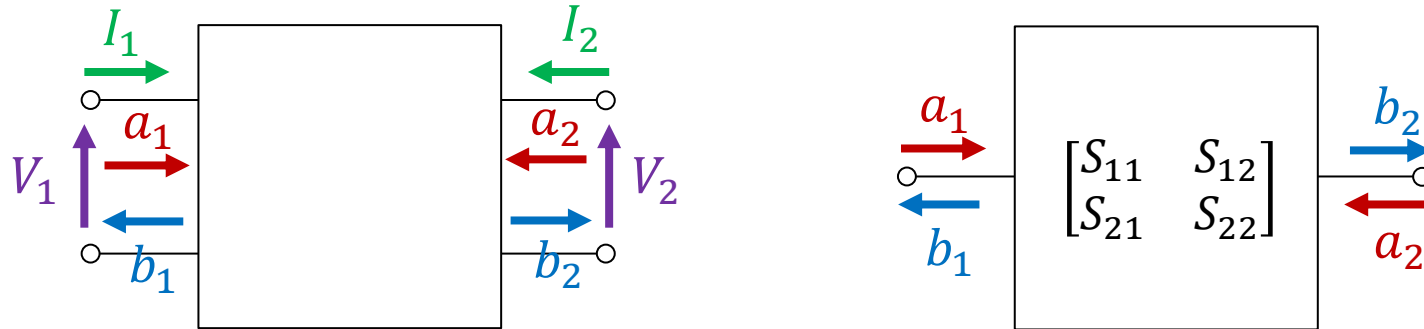
MATLABで確認

```
%% 線路のZ行列
z0=50.;
beta=300.;
len=10.;
z_mat=[-i*z0*cot(beta*len) -i*z0*csc(beta*len);
        -i*z0*csc(beta*len) -i*z0*cot(beta*len)];

%% Z行列⇒S行列への変換
z_mat
z_to_s(z_mat)
s_to_z(z_to_s(z_mat))

%% %%%%%%%%%%% 関数定義 %%%%%%%%%%%
% Z行列⇒S行列への変換
function s = z_to_s(z)
    z0=50.;
    v=[1/sqrt(z0) 0; 0 1/sqrt(z0)];
    s=inv(v*z*v+eye(2))*(v*z*v-eye(2));
end
% S行列⇒Z行列への変換
function z = s_to_z(s)
    z0=50.;
    u=[sqrt(z0) 0; 0 sqrt(z0)];
    z=u*(eye(2)+s)*inv(eye(2)-s)*u;
end
```

T行列



$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

S行列 \Rightarrow T行列

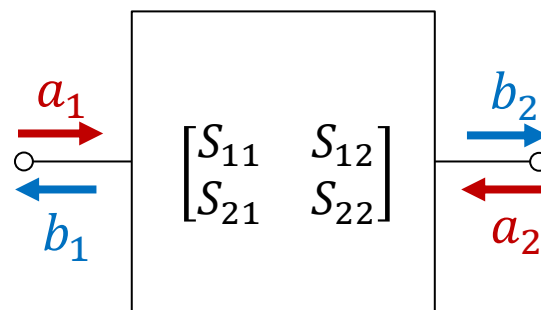
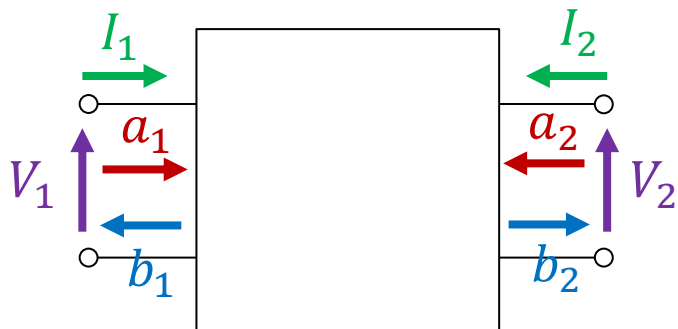
$$\begin{cases} b_1 = S_{11}a_1 + S_{12}a_2 \\ b_2 = S_{21}a_1 + S_{22}a_2 \\ -S_{11}a_1 + b_1 = S_{12}a_2 \\ S_{21}a_1 = b_2 - S_{22}a_2 \end{cases}$$

$$\begin{bmatrix} 1 & -S_{11} \\ 0 & S_{21} \end{bmatrix} \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} S_{12} & 0 \\ -S_{22} & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

T行列は入射波・反射波で表現した場合の縦続接続の計算に便利(F行列は電圧・電流で表現した場合の縦続接続用)。ただし、接続するブロック同士のポート数は同じ出ないとならないので汎用性はあくまで縦続接続した場合に限られる。

$$T = \begin{bmatrix} 1 & -S_{11} \\ 0 & S_{21} \end{bmatrix}^{-1} \begin{bmatrix} S_{12} & 0 \\ -S_{22} & 1 \end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix} S_{12}S_{21} - S_{11}S_{22} & S_{11} \\ -S_{22} & 1 \end{bmatrix}$$

T行列



$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

T行列 \Rightarrow S行列

$$\begin{cases} b_1 = T_{11}a_2 + T_{12}b_2 \\ a_1 = T_{21}a_2 + T_{22}b_2 \end{cases}$$

$$\begin{cases} b_1 - T_{12}b_2 = T_{11}a_2 \\ T_{22}b_2 = a_1 - T_{21}a_2 \end{cases}$$

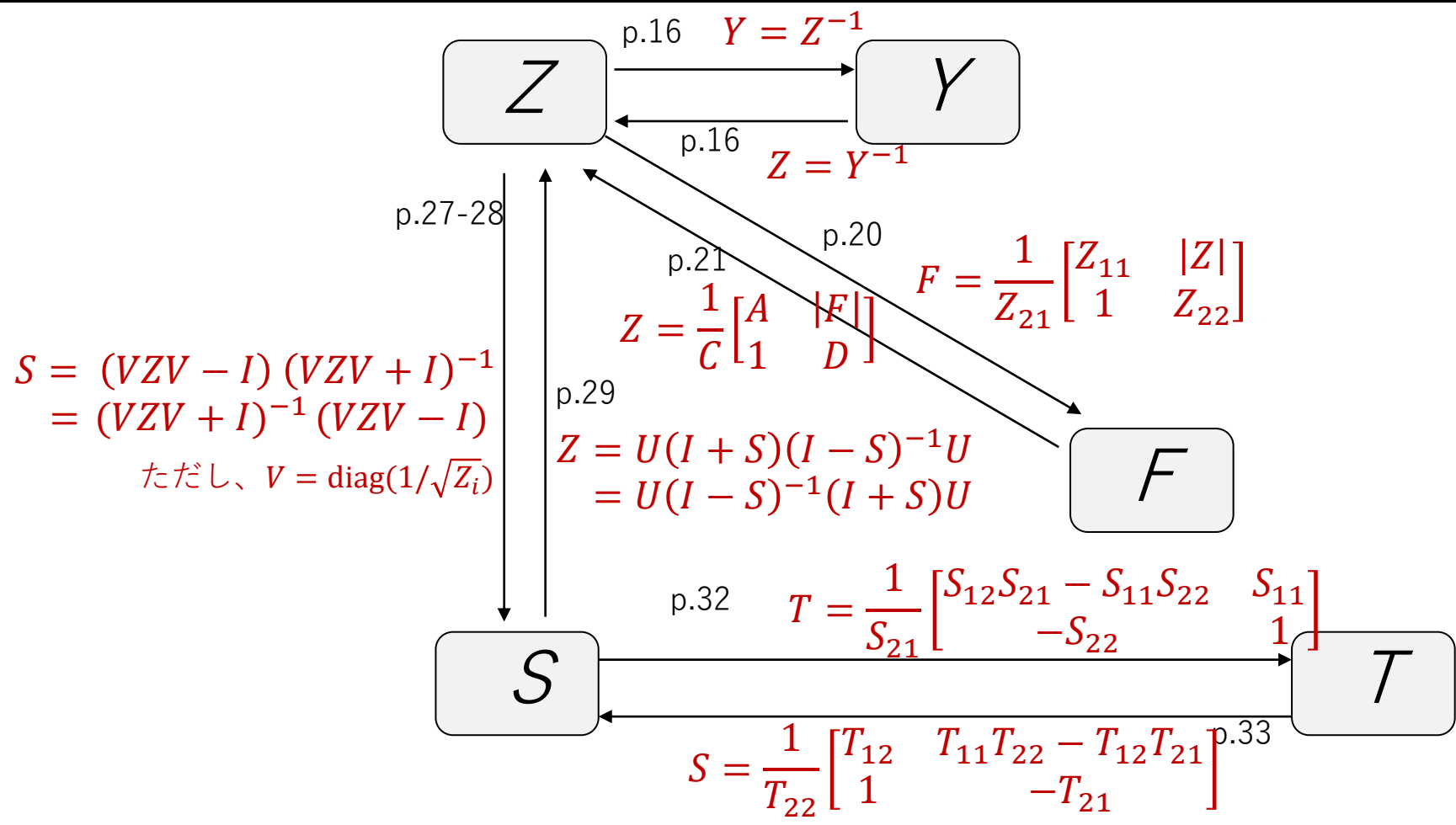
$$\begin{bmatrix} 1 & -T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 & T_{11} \\ 1 & -T_{21} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & -T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 & T_{11} \\ 1 & -T_{21} \end{bmatrix}$$

$$= \frac{1}{T_{22}} \begin{bmatrix} T_{12} & T_{11}T_{22} - T_{12}T_{21} \\ 1 & -T_{21} \end{bmatrix}$$

各種行列の変換



全行列間すべての変換は示していないが（今までの説明のように導出すれば可能である）、これだけでもZ行列を経由して全行列に変換可能である。また、一部にS行列、一部にZ行列を用いるような行列も可能である[1]。

[1] T. Hirano et al. "De-Embedding Method Using an Electromagnetic Simulator for Characterization of Transistors in the Millimeter-Wave Band," IEEE Transactions on Microwave Theory and Techniques, Vol.58, No.10, pp.2663-2672, October 2010.